Lecture 2 : Quantum Integrable Systems

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1 Motivation and definition

Let us recall the classical picture which is at the basis of integrability. First, we are given a real phase space M of dimension $2n$, which we assume for simplicity to admit global Darboux coordinates in terms of n positions q and n momenta p. In convenient situations, we may be able to split M with respect to these two sets of coordinates as $M \simeq M_q \times M_p$. Then, we are also given a Hamiltonian $H(p,q) \in \mathcal{C}^\infty(M)$, which could take the suitable form

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + V\,, \qquad V = V(q) \in \mathcal{C}^\infty(M_q)\,. \tag{1.1}$$

Finally, we have a canonically defined Poisson bracket which allows us to consider the following equations of motion :

$$\frac{df}{dt} = \{f, H\}\,, \tag{1.2}$$

for any function $f \in \mathcal{C}^\infty(M)$ whose evolution we are interested in studying.

The quantum picture that we are going to introduce is motivated by quantum mechanics, and we need to reinterpret the classical construction. We are still considering the space M_q where we can measure positions, but quantum mechanics now tells us that we should be interested in the probability of finding a given system in a particular configuration. We are led to consider a complex Hilbert space $\mathcal{S}$ of states ψ depending on M_q, for example the Hilbert space $L^2(\mathbb{R}^n, dq)$ of complex-valued square-integrable functions on $M_q = \mathbb{R}^n$. What will be important from the point of view of physics is to consider the set $\mathcal{S}_1$ of states of unit norm, such as the unit-length vectors in $L^2(\mathbb{R}^n, dq)$, which can be used to compute the probability of finding a particular system in a region of M_q or to make "measurements". To be more precise, let us introduce an algebra $\mathcal{A}$ of operators, called observables, acting on $\mathcal{S}$. Given an element $\mathcal{F} \in \mathcal{A}$, we can compute the expectation value of $\mathcal{F}$ in a state $\psi \in \mathcal{S}_1$ which is given by

$$\mathrm{E}_\psi[\mathcal{F}] := \langle \psi, \mathcal{F}\psi \rangle\,, \tag{1.3}$$

where $\langle -, - \rangle$ denotes the inner product on $\mathcal{S}$.

A state will evolve over time according to Schrödinger's equation :

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi\,, \quad V = V(q) \in \mathcal{C}^\infty(M_q)\,. \tag{1.4}$$

Here, we use the Laplacian $\Delta f = \sum_{j=1}^{n} \partial^2 f / \partial q_j^2$ for $f \in \mathcal{C}^\infty(M_q)$, while $\hbar$ is the Planck constant. From a more modest point of view, these considerations lead us to the problem of solving (1.4) with

$m = 1$ and $\hbar$ being seen as a formal parameter, from which we get the following simplified version

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi\,, \quad \mathcal{H} = \frac{1}{2}\sum_{j=1}^{n}\left(-\mathrm{i}\hbar\frac{\partial}{\partial q_j}\right)^2 + V\,. \tag{1.5}$$

The element $\mathcal{H}$, called the quantum Hamiltonian, is an element of the algebra $\mathcal{A}$ of operators acting on $\mathcal{S}$.

If the operator $\mathcal{H}$ is time-independent, we can formally solve (1.5) by introducing the one-parameter group of operators

$$\mathcal{U}(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}t\mathcal{H}\right)\,, \quad t \in \mathbb{R}\,. \tag{1.6}$$

It suffices to note that $\mathrm{i}\hbar\frac{\partial \mathcal{U}(t)}{\partial t} = \mathcal{U}(t)\mathcal{H} = \mathcal{H}\mathcal{U}(t)$, in order to get the solution $\psi(t) = \mathcal{U}(t)\psi(0)$ for the initial condition $\psi(0)$. Furthermore, we can use $\mathcal{U}$ to obtain the evolution of an observable $\mathcal{F} \in \mathcal{A}$. Mathematically, we note that[1]

$$\mathrm{E}_{\psi(t)}[\mathcal{F}] = \langle \psi(t), \mathcal{F}\psi(t)\rangle = \langle \psi(0), (\mathcal{U}(t)^{-1}\mathcal{F}\mathcal{U}(t))\psi(0)\rangle = \mathrm{E}_{\psi(0)}[\mathcal{F}(t)]\,, \tag{1.7}$$

for the one-parameter group of operators $\mathcal{F}(t) = \mathcal{U}(t)^{-1}\mathcal{F}\mathcal{U}(t)$. This can be reformulated as saying that the expectation value of the observable $\mathcal{F}$ in the state $\psi(t)$ coincides with the expectation value of the evolution operator $\mathcal{F}(t)$ in the initial state $\psi(0)$. We obtain in that way the Heisenberg picture of dynamics for observables

$$\frac{d\mathcal{F}(t)}{dt} = \frac{-\mathrm{i}}{\hbar}[\mathcal{F}(t), \mathcal{H}]\,, \qquad \mathcal{H} = \frac{1}{2}\sum_{j=1}^{n}\left(-\mathrm{i}\hbar\frac{\partial}{\partial q_j}\right)^2 + V\,, \tag{1.8}$$

where $[-,-]$ denotes the commutator $[A, B] = AB - BA$.

The classical (1.2) and quantum (1.8) equations of motion are quite similar. The role of the Poisson bracket $\{-,-\}$ is now played by the (rescaled) commutator, and both operations are biderivations. Comparing the quantum Hamiltonian $\mathcal{H}$ in (1.5) with its classical counterpart (1.1) suggests that each momentum p_j has its role now being played by the derivation $-\mathrm{i}\hbar\partial/\partial q_j$. Hence we are "replacing" the Poisson algebra $\mathcal{C}^\infty(M)$ by the algebra of operators $\mathcal{A}$ acting on $\mathcal{S}$. Put together, we can give a rough correspondence between the classical and the quantum settings as in Table 1.

In order to understand what is the analogue of an integrable system in the quantum case, let us try to solve (1.5) by decomposing this equation with respect to an eigenbasis for $\mathcal{H}$ of normalised eigenfunctions (ϕ_ℓ) corresponding to a discrete set of complex[2] eigenvalues (E_ℓ), i.e.

$$\phi_\ell \in \mathcal{S}_1 \quad \text{is such that} \quad \mathcal{H}\phi_\ell = E_\ell\phi_\ell\,. \tag{1.9}$$

Then the evolution of some state $\psi(q, 0)$ comes from its decomposition as $\psi(q, 0) = \sum_\ell c_\ell\phi_\ell(q)$ for some constants (c_ℓ). Indeed, it is simply given by

$$\psi(q, t) = \sum_\ell c_\ell\, e^{-\mathrm{i}E_\ell t/\hbar}\phi_\ell(q)\,. \tag{1.10}$$

[1] Here we require that $\mathcal{H}$ is a self-adjoint operator, which means that $\langle \mathcal{H}\phi, \psi\rangle = \langle \phi, \mathcal{H}\psi\rangle$ holds for all $\phi, \psi \in \mathcal{S}$. This implies that $\mathcal{U}(t)$ is unitary, that is $\langle \mathcal{U}(t)\phi, \psi\rangle = \langle \phi, \mathcal{U}(t)^{-1}\psi\rangle$.

[2] The eigenvalues are real if $\mathcal{H}$ is self-adjoint.

	Classical	Quantum	
Poisson algebra of functions	$\mathcal{C}^\infty(M)$	$\mathcal{A}$	Algebra of operators
Classical Hamiltonian	H	$\mathcal{H}$	Quantum Hamiltonian
Poisson bracket	$\{-,-\}$	$\frac{-\mathrm{i}}{\hbar}[-,-]$	Commutator
Classical equation of motion	$\frac{df}{dt}=\{f,H\}$	$\frac{d\mathcal{F}}{dt}=\frac{-\mathrm{i}}{\hbar}[\mathcal{F},\mathcal{H}]$	Heisenberg equation

Table 1: correspondence between classical and quantum notions.

So now the question reduces to find an eigenbasis for $\mathcal{H}$. By definition, an operator $\mathcal{F}\in\mathcal{A}$ commutes with $\mathcal{H}$ if

$$[\mathcal{F},\mathcal{H}]=\mathcal{F}\mathcal{H}-\mathcal{H}\mathcal{F}=0\,. \tag{1.11}$$

The key remark is to realise that if $\mathcal{F}$ commutes with $\mathcal{H}$, then

$$\mathcal{H}\mathcal{F}\phi_\ell=\mathcal{F}\mathcal{H}\phi_\ell=E_\ell\mathcal{F}\phi_\ell\,, \tag{1.12}$$

so that $\mathcal{F}\phi_\ell$ is itself an eigenvector of $\mathcal{H}$ of eigenvalue E_ℓ. Therefore, we can consider the more precise problem of finding a common normalised eigenbasis of $\mathcal{H}$ and $\mathcal{F}$, that is we want

$$\phi_\ell\in\mathcal{S}_1 \quad \text{such that } \mathcal{H}\phi_\ell=E_\ell\phi_\ell,\ \mathcal{F}\phi_\ell=E'_\ell\phi_\ell,\ \ E_\ell,E'_\ell\in\mathbb{C}\,. \tag{1.13}$$

This strategy can be repeated with an operator commuting with both $\mathcal{H}$ and $\mathcal{F}$, then a fourth operator commuting with the first three ones, and so on and so forth. But when should we stop? In the classical case, we limited our choice of Poisson commuting elements to $n=\frac{1}{2}\dim(M)=\dim(M_q)$ functions which are functionally independent, so we shall introduce a similar property of independence.

Definition 1.1. *A set of operators $\mathcal{H}_1,\ldots,\mathcal{H}_n\in\mathcal{A}$ is a* quantum (completely) integrable system *if these operators pairwise commute (i.e. $[\mathcal{H}_i,\mathcal{H}_j]=0$ for all $1\le i,j\le n$) and there exists no relation between them.*

A word of warning here : the existence of a relation usually means that we can write the identity

$$P(\mathcal{H}_1,\ldots,\mathcal{H}_n)=0\,, \tag{1.14}$$

for some nonzero polynomial $P\in\mathbb{C}[z_1,\ldots,z_n]$ in n variables. (Since the operators $(\mathcal{H}_i)$ pairwise commute, replacing the variables (z_i) by them yields a well-defined operator.) Depending on the context, P may have a slightly different form, e.g. we could seek independence over real-algebraic/analytic functions in n variables. As an example, note that for $i\neq j$ there exists a relation between the three commuting operators $\mathcal{F}_1=\partial^2/\partial q_i^2$, $\mathcal{F}_2=\partial^2/\partial q_j^2$, and $\mathcal{F}_3=\partial/\partial q_i+\partial/\partial q_j$, since

$$\text{for} \quad P(z_1,z_2,z_3)=(z_3^2-z_1-z_2)^2-4z_1z_2\,, \quad \text{we have} \quad P\left(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\right)=0\,. \tag{1.15}$$

There is a widespread use of a more convenient definition of quantum integrability which is motivated by the classical case, and it has the advantage of getting the independence of the operators "for free". This is based on the method of deformation/quantization. The quantization process consists in making the formal replacement

$$p_j \ \rightsquigarrow\ -\mathrm{i}\hbar\frac{\partial}{\partial q_j}\,, \tag{1.16}$$

to go from a polynomial function on M to an operator belonging to $\mathcal{A}$. There is an obvious issue of ordering here, since the function $q_j p_j$ can be quantized in the many forms

$$\alpha q_j p_j + (1-\alpha) p_j q_j \ \rightsquigarrow\ -\alpha \mathrm{i}\hbar q_j \frac{\partial}{\partial q_j} - (1-\alpha)\mathrm{i}\hbar \frac{\partial}{\partial q_j} q_j = -\mathrm{i}\hbar q_j \frac{\partial}{\partial q_j} + (\alpha - 1)\mathrm{i}\hbar\,, \quad \alpha \in \mathbb{C}\,.$$

Nevertheless, it is well-defined for the Hamiltonian operators of interest since

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + V(q) \ \rightsquigarrow\ \mathcal{H} = \frac{1}{2}\sum_{j=1}^{n}\left(-\mathrm{i}\hbar\frac{\partial}{\partial q_j}\right)^2 + V(q)\,. \tag{1.17}$$

The "opposite" of the map (1.16) is given by the replacement

$$-\mathrm{i}\hbar\frac{\partial}{\partial q_j} \ \rightsquigarrow\ p_j\,, \qquad \text{then} \qquad \hbar \to 0\,, \tag{1.18}$$

and it is called the reduction. For example, H is the reduction of $\mathcal{H}$ in (1.17). Note that performing quantization (1.16) then reduction (1.18) returns the original function, but the opposite may not be true. With this set up, we have the following alternative definition of quantum integrability.

Definition 1.2 (Alternative to Definition 1.1)**.** *A set of operators $\mathcal{H}_1, \dots, \mathcal{H}_n \in \mathcal{A}$ is a* quantum (completely) integrable system *if these operators pairwise commute ($[\mathcal{H}_i, \mathcal{H}_j] = 0$) and their reductions $H_1, \dots, H_n$ are functionally independent.*

2 Examples of quantum integrable systems

Remark 2.1. *To ease notations when dealing with examples, it is usual to fix $\hbar$ to some value in $\mathbb{C} \setminus \{0\}$. In our examples, we will work with $\hbar = -\mathrm{i}$ so that the Hamiltonian operators are of the form $\mathcal{H} = \frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial q_j^2} + V$, and quantization is just $p_j \rightsquigarrow \frac{\partial}{\partial q_j}$.*

2.1 The harmonic oscillator

Let us start with the Hamiltonian of the classical harmonic oscillator

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \omega^2 \sum_{j=1}^{n} q_j^2\,, \quad \omega \in \mathbb{R}\setminus\{0\}\,, \tag{2.1}$$

which is defined on the space $M = T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$. The Poisson bracket is taken to be the canonical one. We note that the Hamiltonian can be uncoupled in terms of its $n = 1$ subcases as

$$H = \sum_{j=1}^{n} h_j\,, \quad h_j := \frac{1}{2}p_j^2 + \omega^2 q_j^2\,, \tag{2.2}$$

and we easily see that the n functions from $h := (h_j)_{1\le j\le n}$ are Poisson commuting. Furthermore, we can compute the Jacobian matrix as

$$J_h = \begin{pmatrix} \frac{\partial h}{\partial q} & \frac{\partial h}{\partial p}\end{pmatrix} = \left(\begin{array}{ccc|ccc} 2\omega^2 q_1 & & & p_1 & & \\ & \ddots & & & \ddots & \\ & & 2\omega^2 q_n & & & p_n \end{array}\right). \tag{2.3}$$

It is easy to see that each diagonal $n \times n$ block submatrix of J_h is invertible on a dense subset of the phase space M, hence the functions in h are functionally independent. It then follows that the functions

$$H_1 := H,\ H_2 := h_2,\ \dots\ ,\ H_n := h_n \tag{2.4}$$

define an integrable system, so that the Hamiltonian H is integrable.

The quantization of the Hamiltonian (2.1) is uniquely defined and is just

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2} + \omega^2 \sum_{j=1}^{n} q_j^2\,. \tag{2.5}$$

Here, we see $\mathcal{H}$ as an operator on the space of states $\mathcal{S} = L^2(\mathbb{R}^n, dq)$, which contains the square-integrable functions defined on $\mathbb{R}^n$. (To get $\mathcal{S}_1$, we would restrict further to the states which have unit length.) The quantization of the functions from the integrable system (2.4) are given by

$$\mathcal{H}_1 := H,\ \mathcal{H}_2 := \frac{1}{2}\frac{\partial^2}{\partial q_2^2} + \omega^2 q_2^2,\ \dots\ ,\ \mathcal{H}_n := \frac{1}{2}\frac{\partial^2}{\partial q_n^2} + \omega^2 q_n^2\,, \tag{2.6}$$

and it is straightforward to see that such operators pairwise commute. If we are back to our original problem of finding solutions to the Schrödinger's equation (1.4), recall that we want to find a common eigenbasis of the operators (2.6). The quantum integrable system tells us that we have to look at the solutions of the $n = 1$ cases

$$\left(\frac{1}{2}\frac{\partial^2}{\partial q_j^2} + \omega^2 q_j^2\right) \phi_\ell(q_j) = E_{\ell,j}\phi_\ell(q_j)\,, \quad \ell \in \mathbb{N}\,, \tag{2.7}$$

which can be expressed using Hermite polynomials, see Exercise 3.3. Thus, we can use these functions to build eigenfunctions of $\mathcal{H}$ such as

$$\phi_\lambda(q) := \phi_{\ell_1}(q_1)\dots\phi_{\ell_n}(q_n)\,, \quad \lambda = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n\,. \tag{2.8}$$

2.2 A Lax pair for the Calogero-Moser system

We begin with a system describing the motion of n interacting particles on the real line with positions (q_j). The system would have the general form

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n} V(q_k - q_j)\,, \tag{2.9}$$

for some potential function V. We will be interested in the case of the rational potential

$$V(x) = g^2\frac{1}{x^2}\,, \quad g > 0\,, \tag{2.10}$$

which is associated to an important integrable system, called the Calogero-Moser system. (Note that it admits several generalisations that bear the same name but which will not be presented in these introductory notes.) As a first step towards this system, let us note the observation made by Moser that if we consider the matrices

$$L \in \mathrm{Mat}_{n\times n}(\mathbb{C})\,, \quad L_{jj} = p_j, \quad L_{jk} = \frac{\mathrm{i}g}{q_j - q_k}, \quad j \neq k\,, \tag{2.11a}$$

$$M \in \mathrm{Mat}_{n\times n}(\mathbb{C})\,, \quad M_{jj} = \sum_{k\neq j} \frac{\mathrm{i}g}{(q_j - q_k)^2}, \quad M_{jk} = \frac{-\mathrm{i}g}{(q_j - q_k)^2}\,, \quad j \neq k\,, \tag{2.11b}$$

then we have in fact a Lax pair. To see this property, we consider the Hamiltonian vector field $\frac{d}{dt} = \frac{1}{2}\{-, \operatorname{tr} L^2\}$, and we remark that we can write

$$\frac{dL}{dt} = [L, M] \,. \tag{2.12}$$

This is a matrix identity, i.e. $dL_{jk}/dt = (LM)_{jk} - (ML)_{jk}$ for all j,k, which can be checked in Exercise 3.4. As we learnt in Lecture 1, having a Lax pair guarantees that the functions

$$H_1 = \operatorname{tr} L,\ H_2 = \frac{1}{2}\operatorname{tr} L^2,\ \ldots\ ,\ H_n = \frac{1}{n}\operatorname{tr} L^n, \tag{2.13}$$

are first integrals of

$$H_{CM} := H_2 = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n} \frac{g^2}{(q_j-q_k)^2} \,. \tag{2.14}$$

The function H_{CM} has the form (2.9) and it is the Hamiltonian of the Calogero-Moser system.

We can build an integrable system by considering the functions (2.13), and we will sketch two ways to derive this result in the next subsection. For later use, let us note that the matrix L from (2.11a) is a Hermitian matrix, i.e. $\bar{L}_{kj} = L_{jk}$. If we introduce

$$Q \in \operatorname{Mat}_{n\times n}(\mathbb{C}) \,, \quad Q = \operatorname{diag}(q_1, \ldots, q_n) \,, \tag{2.15}$$

which is also Hermitian, the pair (Q, L) satisfies

$$QL - LQ = -\mathrm{i}g(\operatorname{Id}_n - vv^\dagger) \,, \quad v = (1, \ldots, 1)^T \,. \tag{2.16}$$

(Here $A^\dagger$ denotes the conjugate transpose of a matrix A. In the simple case at hand $v^\dagger = (1, \ldots, 1)$.)

To get a quantum version of the Calogero-Moser system, let us first look at the Hamiltonian (2.14). Its quantization under (1.16) is unique, and it can be written as

$$\mathcal{H}_{CM} = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2} + \sum_{1\leq j<k\leq n} \frac{g^2}{(q_j-q_k)^2} \,. \tag{2.17}$$

In order to find a family of commuting operators containing $\mathcal{H}_{CM}$, let us simply quantize the Lax pair (L, M). This can be done uniquely using the matrix-valued operators $(\mathcal{L}, \mathcal{M})$ defined by

$$\mathcal{L}_{jj} = \frac{\partial}{\partial q_j}, \quad \mathcal{L}_{jk} = \frac{\mathrm{i}a}{q_j - q_k}, \quad j \neq k \,, \tag{2.18a}$$

$$\mathcal{M}_{jj} = \sum_{k\neq j} \frac{\mathrm{i}a}{(q_j-q_k)^2}, \quad \mathcal{M}_{jk} = \frac{-\mathrm{i}a}{(q_j-q_k)^2} \,, \quad j \neq k \,. \tag{2.18b}$$

Note that we introduce a constant $a \in \mathbb{C}^\times$ in those expressions satisfying $g^2 = a^2 - \mathrm{i}a$[3]. Let us also remark that the functions appearing in these matrices are seen as operators, so for example

$$\frac{\partial}{\partial q_j}\,\mathcal{L}_{jk} = \mathcal{L}_{jk}\,\frac{\partial}{\partial q_j} - \frac{\mathrm{i}a}{(q_j-q_k)^2} \,,$$

[3]This odd-looking condition is due to our choice for $\hbar$ made in Remark 2.1. In full generalities, we replace $\mathrm{i}a$ by $\mathrm{i}g$ and $-\mathrm{i}a$ by $\hbar g$ in $\mathcal{L}$ and $\mathcal{M}$ respectively. Then the role of the element $a^2 - \mathrm{i}a$ will be played by $g(g+\hbar)$ which tends to g^2 for $\hbar \to 0$.

because this identity holds when applied to a function by the chain rule. We can then use the pair (2.18a)–(2.18b) to note that under the derivation $\frac{d}{dt} = [\mathcal{H}_{CM}, -]$, we can write

$$\frac{d\mathcal{L}}{dt} = [\mathcal{L}, \mathcal{M}]\,, \tag{2.19}$$

which looks similar to (2.12). Let us dissect (2.19), which is an identity of matrix-valued operators. On one side, the (j,k) entry is given by the commutator of operators $[\mathcal{H}_{CM}, \mathcal{L}_{jk}]$, while on the other side we have the entry $\sum_{s=1}^{n}(\mathcal{L}_{js}\mathcal{M}_{sk} - \mathcal{M}_{js}\mathcal{L}_{ks})$ of a commutator of matrices.

We can now use (2.19) to find first integrals of $\mathcal{H}_{CM}$. A first attempt based on the classical case is to look at traces of powers of $\mathcal{L}$, and we find

$$\frac{d\,\mathrm{tr}\,\mathcal{L}^m}{dt} = \sum_{\mu=0}^{m-1} \mathrm{tr}\left(\mathcal{L}^\mu \frac{d\mathcal{L}}{dt} \mathcal{L}^{m-\mu-1}\right) = \mathrm{tr}([\mathcal{L}^m, \mathcal{M}])\,, \quad m \geq 1\,. \tag{2.20}$$

Unfortunately, this does not necessarily vanish : we are in presence of operators so that $\mathrm{tr}(\mathcal{L}^m\mathcal{M}) \neq \mathrm{tr}(\mathcal{M}\mathcal{L}^m)$ in general. So we can not simply use traces of powers of the quantum Lax matrix $\mathcal{L}$ (2.18a) as operators commuting with $\mathcal{H}_{CM}$. To be successful, we need in fact to reformulate the first integrals found in the classical case. Namely, we note in view of (2.16) that

$$0 = \mathrm{tr}\left((QL - LQ)L^m\right) = -ig\,\mathrm{tr}\left((\mathrm{Id}_n - vv^\dagger)L^m\right)\,, \quad v = (1, \dots, 1)^T\,, \tag{2.21}$$

and we can write in particular

$$H_1 = v^\dagger L v,\ H_2 = \frac{1}{2} v^\dagger L^2 v,\ \dots\ ,\ H_n = \frac{1}{n} v^\dagger L^n v. \tag{2.22}$$

Trying these alternative forms in the quantum case, we have

$$\frac{dv^\dagger \mathcal{L}^m v}{dt} = v^\dagger [\mathcal{L}^m, \mathcal{M}] v = \sum_{i,j,k=1}^{n} \mathcal{L}^m_{ij} \mathcal{M}_{jk} - \sum_{i,j,k=1}^{n} \mathcal{M}_{ij} \mathcal{L}^m_{jk}\,. \tag{2.23}$$

We now observe that this expression vanishes in view of the identities

$$\sum_{k=1}^{n} \mathcal{M}_{jk} = 0 = \sum_{k=1}^{n} \mathcal{M}_{kj}\,, \quad \text{for all } j = 1, \dots, n\,. \tag{2.24}$$

As a corollary, the operators

$$\mathcal{H}_1 = v^\dagger \mathcal{L} v,\ \mathcal{H}_2 = \frac{1}{2} v^\dagger \mathcal{L}^2 v,\ \dots\ ,\ \mathcal{H}_n = \frac{1}{n} v^\dagger \mathcal{L}^n v\,, \tag{2.25}$$

are all commuting with $\mathcal{H}_{CM}$, which is in fact equal to $\mathcal{H}_2$. Furthermore, it can be shown that these operators are pairwise commuting, and we will prove this result in § 2.3.3. Their independence follows from the classical case since the reduction of the matrix $\mathcal{L}$ (2.18a) is given by the matrix L (2.11a) from the classical case. (We refer to Footnote 3 for the precise way to reintroduce $\hbar$, so that we can take the limit $\hbar \to 0$ in the reduction.)

2.3 More on the Calogero-Moser system

2.3.1 Hamiltonian reduction

Note that if L from (2.11a) can be diagonalised into $\tilde{L}$ which has distinct eigenvalues, then we can choose a unitary matrix $U \in \mathrm{U}(n)$ such that $L = U^{-1}\tilde{L}U$ and $Uv = v$ for $v = (1, \ldots, 1)^T$, see Exercise 3.7. The pair

$$(\tilde{Q}, \tilde{L}) = (UQU^{-1}, ULU^{-1}) \tag{2.26}$$

satisfies (2.16), and as a corollary we can write their entries as

$$\tilde{L}_{jk} = \delta_{jk}\tilde{p}_j\,, \quad \tilde{Q}_{jk} = \delta_{jk}\tilde{q}_j + (1-\delta_{jk})\frac{-\mathrm{i}g}{\tilde{p}_j - \tilde{p}_k}\,, \tag{2.27}$$

for some $(\tilde{q}_j, \tilde{p}_j)$ which are another set of coordinates. By invariance of the trace, we obtain in those coordinates that

$$H_1 = \operatorname{tr}\tilde{L} = \sum_{1\le j\le n}\tilde{p}_j,\ \ldots\ ,\ H_n = \frac{1}{n}\operatorname{tr}\tilde{L}^n = \frac{1}{n}\sum_{1\le j\le n}\tilde{p}_j^n\,. \tag{2.28}$$

The independence of those functions is obtained as follows : their Jacobian matrix taken with respect to the coordinates $(\tilde{p}_j)$ is the Vandermonde matrix $V = (V_{jk})$, $V_{jk} = \tilde{q}_k^{j-1}$, whose determinant is nonzero whenever the $(\tilde{p}_j)$ are pairwise distinct. In fact, it is possible to compute that the Poisson bracket of these alternative coordinates is given by

$$\{\tilde{q}_j, \tilde{q}_k\} = 0 = \{\tilde{p}_j, \tilde{p}_k\}\,, \quad \{\tilde{q}_j, \tilde{p}_k\} = \delta_{jk}\,, \tag{2.29}$$

hence the functions (2.13) are trivially Poisson commuting, and thus form an integrable system.

What we have sketched is a rough application of Hamiltonian reduction, which is a very useful tool within the field of integrable systems to obtain examples from spaces of matrices. In full generalities, we start with the space N of pairs of Hermitian matrices (X, Y), endowed with the non-degenerate Poisson bracket defined by

$$\{X_{ij}, Y_{kl}\} = \delta_{kj}\delta_{il}\,, \quad \{X_{ij}, X_{kl}\} = 0\,, \quad \{Y_{ij}, Y_{kl}\} = 0\,. \tag{2.30}$$

We then consider the subspace N_g obtained by imposing the (moment map) condition that the Hermitian matrix $\mathrm{i}[X,Y]$ has value in the $\mathrm{U}(n)$ coadjoint orbit of elements of the form $-g\,\mathrm{Id}_n + ww^\dagger$, for $w \in \mathbb{C}^n$ nonzero. Finally, we consider the associated orbit space $N_g/\,\mathrm{U}(n)$. We can see that if X (resp. Y) is diagonalisable with distinct eigenvalues, the orbit of the pair (X, Y) contains an element of the form (Q, L) (resp. $(\tilde{Q}, \tilde{L})$). An important byproduct of this method is that the flows of the Hamiltonian vector field associated to $\frac{1}{k}\operatorname{tr}Y^k$ on N are simply given by $(X_0, Y_0) \mapsto (X_0 + tY_0^{k-1}, Y_0)$. We can thus understand the (reduced) flows under H_k in terms of the element (q_j, p_j) by projection of the flows from N onto $N_g/\,\mathrm{U}(n)$, in which we select an element of the orbit in the form (Q, L).

2.3.2 Integrability from an r-matrix

Let us introduce for any $1 \le i, j \le n$ the elementary matrix E_{ij} whose only nonzero entry is given by $+1$ in position (i, j). Then, we can write an arbitrary matrix as $A = \sum_{j,k} A_{jk}\,E_{jk}$. For example, the Lax matrix L from (2.11a) takes the simple form

$$L = \sum_{j=1}^{n} p_j E_{jj} + \sum_{j\neq k}\frac{\mathrm{i}g}{q_j - q_k}E_{jk}\,. \tag{2.31}$$

In the same way, we can decompose a tensor product of matrices as

$$r = \sum_{ijkl=1}^{n} r_{ij,kl}\, E_{ij} \otimes E_{kl}\,. \tag{2.32}$$

(We consider unadorned tensor products over $\mathbb{C}$.) If we are given an element r of the form (2.32), it is a standard notation to denote $r_{12} = r$, and we introduce the following element obtained by permuting the two copies in the tensor product :

$$r_{21} = \sum_{ijkl=1}^{n} r_{ij,kl}\, E_{kl} \otimes E_{ij}\,. \tag{2.33}$$

Next, let us see an arbitrary matrix A as an element of the form (2.32) in two different ways:

$$A_1 = A \otimes \mathrm{Id}_n = \sum_{ijkl=1}^{n} A_{ij}\delta_{kl}\, E_{ij} \otimes E_{kl}\,, \quad A_2 = \mathrm{Id}_n \otimes A = \sum_{ijkl=1}^{n} \delta_{ij} A_{kl}\, E_{ij} \otimes E_{kl}\,. \tag{2.34}$$

Finally, if two matrices A, B are seen as functions over a phase space endowed with a Poisson bracket, we introduce the notation

$$\{A_1 \overset{\otimes}{,} B_2\} = \sum_{ijkl=1}^{n} \{A_{ij}, B_{kl}\}\, E_{ij} \otimes E_{kl}\,. \tag{2.35}$$

We are now able to state a strong result, due to Babelon and Viallet. A matrix L defined over a phase space is such that its symmetric functions ($\mathrm{tr}\, L^k$) are all pairwise commuting, if and only if there exists some element r of the form (2.32) such that[4]

$$\{L_1 \overset{\otimes}{,} L_2\} = [r_{12}, L_1] - [r_{21}, L_2]\,. \tag{2.36}$$

Let us prove the "only if" part. We will use the notation tr_2 to mean that we take the trace with respect to the second factor in the tensor product for an expression of the form (2.32). We first note that for any $m \geq 1$,

$$\frac{1}{m}\{L, \mathrm{tr}(L^m)\} = \sum_{ijkl} \{L_{ij}, L_{kl}\} L_{lk}^{m-1}\, E_{ij} = \mathrm{tr}_2 \left(\{L_{ij} \overset{\otimes}{,} L_2\} L_2^{m-1} \right). \tag{2.37}$$

Upon substituting (2.36), we find

$$\frac{1}{m}\{L, \mathrm{tr}(L^m)\} = \mathrm{tr}_2 \left([r_{12}, L_1] L_2^{m-1} \right) - \mathrm{tr}_2 \left(r_{21} L_2^m - L_2^{m-1} r_{21} L_2 \right), \tag{2.38}$$

and the second term vanishes by cyclicity of the trace. If we note that $M_m = -\,\mathrm{tr}_2 \left(r_{12} L_2^{m-1} \right)$ is a matrix, we have

$$\frac{1}{m}\{L, \mathrm{tr}(L^m)\} = [L, M_m]\,. \tag{2.39}$$

In other words, we have a Lax pair (L, M_m) for the Hamiltonian vector field associated to each function $\frac{1}{m}\,\mathrm{tr}(L^m)$, hence these functions are all Poisson commuting. The upshot of this discussion

[4] Here, $[-,-]$ is simply the commutator of matrices. In particular $[A' \otimes A'', B' \otimes B''] = A'B' \otimes A''B'' - B'A' \otimes B''A''$.

is that we can use the result of Babelon and Viallet in the case of the Calogero-Moser system. Indeed, if we introduce

$$r_{12} = a_{12} + b_{12}\,, \quad a_{12} = \sum_{i\neq j} \frac{1}{q_i - q_j} E_{ij} \otimes E_{ji}\,, \quad b_{12} = \sum_{i\neq j} \frac{1}{q_i - q_j} E_{ii} \otimes E_{ij}\,, \tag{2.40}$$

then the matrix L from (2.11a) and the element r_{12} from (2.40) satisfy (2.36), see Exercise 3.8. In particular, the different functions in (2.13) are Poisson commuting, as we expected.

Let us now turn to a simple question : what is so special about the matrix r given in (2.40), or more generally such a matrix satisfying (2.36)? The answer lies in writing Jacobi identity in tensor notations. If we work with a tensor product of three copies of the algebra of matrix-valued functions, an element r of the form (2.32) can be extended as $r_{12} = r \otimes \mathrm{Id}_n$, and we can also introduce

$$r_{13} = \sum_{ijkl=1}^{n} r_{ij,kl}\, E_{ij} \otimes \mathrm{Id}_n \otimes E_{kl}\,, \quad r_{23} = \sum_{ijkl=1}^{n} r_{ij,kl}\, \mathrm{Id}_n \otimes E_{ij} \otimes E_{kl}\,, \tag{2.41}$$

or define r_{21}, r_{31}, r_{32} in the same way. Given a matrix L, we introduce $L_1 = L \otimes \mathrm{Id}_n \otimes \mathrm{Id}_n$ and then define L_2, L_3 in a similar way. A key property of (2.36) is that Jacobi identity written as

$$\{L_1 \overset{\otimes}{,} \{L_2 \overset{\otimes}{,} L_3\}\} + \text{ cyclic permutations } = 0\,, \tag{2.42}$$

can be recast as

$$[L_1,\, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2 \overset{\otimes}{,} r_{13}\} - \{L_3 \overset{\otimes}{,} r_{12}\}] + \text{ cyc. perm. } = 0\,. \tag{2.43}$$

Therefore, if we attempt to classify all the pairs (L, r) such that (2.36) holds, we are also interested in this auxiliary identity. If r is a constant (which is not the case for (2.40)), we can get a solution to (2.43) in the simpler situation where

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0\,. \tag{2.44}$$

This identity is called the *classical Yang-Baxter equation*[5]. We call a solution of (2.44) an r-matrix, and from our discussion we see that r-matrices can be particularly interesting to study.

We will not adapt the above approach to the quantum setting, as it would become rather involved. Let us simply mention that the quantum version of (2.44) is written as

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,, \tag{2.45}$$

and is called the *quantum Yang-Baxter equation*. If we can expand a solution R of the quantum Yang-Baxter equation as $R = \mathrm{Id}_n - \hbar r + o(\hbar^2)$, then the linear part r will satisfy (2.44). Such solutions have a deep connection to quantum groups and to various versions of integrability in the quantum setting, which will not be touched as part of this lecture series.

2.3.3 The quantum first integrals commute

We have obtained that the operators $\mathcal{H}_k$ defined in (2.25) are commuting with the Calogero-Moser operator $\mathcal{H}_{CM} = \mathcal{H}_2$ thanks to the Lax pair (2.19). We can also show that these operators commute with $\mathcal{H}_1 = \sum_j \frac{\partial}{\partial q_j}$ in view of the simple equality

$$[\mathcal{H}_1, \mathcal{L}] = 0\,. \tag{2.46}$$

[5]It is also usual to keep that name for an element r satisfying the antisymmetry property $r_{12} = -r_{21}$.

To prove the identity $[\mathcal{H}_k, \mathcal{H}_m] = 0$ for arbitrary $k, m \geq 1$, there is still some work to do. First, we compute the third Hamiltonian operator

$$\mathcal{H}_3 = \frac{1}{3}\sum_{j=1}^{n} \frac{\partial^3}{\partial q_j^3} + \frac{g^2}{3}\sum_{\substack{k,j=1\\k\neq j}}^{n}\left[\frac{\partial}{\partial q_j}\frac{1}{(q_j-q_k)^2} + \frac{1}{(q_j-q_k)}\frac{\partial}{\partial q_k}\frac{1}{(q_j-q_k)} + \frac{1}{(q_j-q_k)^2}\frac{\partial}{\partial q_j}\right]\,. \tag{2.47}$$

(Here, we used that $g^2 = a^2 - \mathrm{i}a$.) This identity allows us to prove that

$$[\mathcal{H}_3, \mathcal{L}] = [\mathcal{L}, \mathcal{M}^{(3)}]\,, \tag{2.48a}$$

$$\text{where } \mathcal{M}^{(3)}_{jk} = (1-\delta_{jk})\mathcal{N}_{jk} - \delta_{jk}\sum_{l\neq j}\mathcal{N}_{jl}\,, \tag{2.48b}$$

$$\text{with } \mathcal{N}_{jk} := \frac{-\mathrm{i}a}{(q_j-q_k)^2}\left(\frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_k}\right) + a^2\sum_{l\neq j,k}\frac{1}{(q_j-q_l)(q_l-q_k)(q_j-q_k)},\ j\neq k\,. \tag{2.48c}$$

We directly get $\sum_k \mathcal{M}^{(3)}_{jk} = 0$ for any j, and after a bit of manipulations we can also find that $\sum_j \mathcal{M}^{(3)}_{jk} = 0$. We then deduce from these two identities and (2.48a) that $[\mathcal{H}_3, \mathcal{H}_m] = 0$ for all $m \geq 1$, see Exercise 3.14. Hence, we have been able to prove that when $k = 1, 2, 3$,

$$[\mathcal{H}_k, \mathcal{H}_m] = 0 \quad \text{for all } m \geq 1\,. \tag{2.49}$$

We now proceed to obtain this result for arbitrary k by induction. To this end, we introduce the operator

$$\mathcal{J}_2 = \frac{1}{2}\sum_{j=1}^{n}[q_j^2, \mathcal{H}_3]\,, \tag{2.50}$$

which satisfies the identity

$$[\mathcal{J}_2, \mathcal{H}_m] = (m+1)\mathcal{H}_{m+1}\,, \qquad \text{for } m \geq 1\,. \tag{2.51}$$

We will check (2.51) shortly, but before that let us derive the long-awaited result of commutativity of the operators $(\mathcal{H}_k)$. Assuming that (2.49) holds for some k (which we already know for $k = 1, 2, 3$), we use (2.50) and Jacobi identity for the commutator to get

$$\begin{aligned}\left[\mathcal{H}_{k+1}, \mathcal{H}_m\right] &= \frac{1}{k+1}[[\mathcal{J}_2, \mathcal{H}_k], \mathcal{H}_m] = \frac{-1}{k+1}\left([[\mathcal{H}_m, \mathcal{J}_2], \mathcal{H}_k] + [[\mathcal{H}_k, \mathcal{H}_m], \mathcal{J}_2]\right)\\ &= \frac{m+1}{k+1}[\mathcal{H}_{m+1}, \mathcal{H}_k] - \frac{1}{k+1}[[\mathcal{H}_k, \mathcal{H}_m], \mathcal{J}_2]\,.\end{aligned} \tag{2.52}$$

By the inductive hypothesis, $[\mathcal{H}_{m+1}, \mathcal{H}_k] = 0 = [\mathcal{H}_k, \mathcal{H}_m]$ so that $[\mathcal{H}_{k+1}, \mathcal{H}_m] = 0$ for any $m \geq 1$. Hence we get $[\mathcal{H}_k, \mathcal{H}_m] = 0$ for all indices, which proves that the operators (2.25) commute.

We now turn to prove (2.51), which relies on the Lax pair $(\mathcal{L}, \mathcal{M}^{(3)})$ that we derived in (2.48). Since we got from (2.48) that $[\mathcal{H}_3, \mathcal{H}_m] = 0$ for any $m \geq 1$, Jacobi identity yields

$$[\mathcal{J}_2, \mathcal{H}_m] = \frac{1}{2}\sum_{j=1}^{n}[[q_j^2, \mathcal{H}_3], \mathcal{H}_m] = -\frac{1}{2}\sum_{j=1}^{n}[[\mathcal{H}_m, q_j^2], \mathcal{H}_3]\,. \tag{2.53}$$

Noting that $[\mathcal{L}_{kl}, q_j] = \delta_{kl}\delta_{kj}$, we can write

$$\begin{aligned}\left[\mathcal{J}_2, \mathcal{H}_m\right] =& \frac{-1}{2m}\sum_{\mu=0}^{m-1}\sum_{jklk'l'}\left[(\mathcal{L}^\mu)_{kk'}[\mathcal{L}_{k'l'}, q_j^2](\mathcal{L}^{m-\mu-1})_{l'l}, \mathcal{H}_3\right] \\ =& \frac{-1}{m}\sum_{\mu=0}^{m-1}\sum_{jkl}[(\mathcal{L}^\mu)_{kj} q_j (\mathcal{L}^{m-\mu-1})_{jl}, \mathcal{H}_3]\,.\end{aligned} \tag{2.54}$$

In view of (2.48a) and the fact that the commutator satisfies Leibniz's rule, we get for any $s \geq 1$ the identity $[\mathcal{L}^s, \mathcal{H}_3] = [\mathcal{M}^{(3)}, \mathcal{L}^s]$. Thus

$$\begin{aligned}\left[\mathcal{J}_2, \mathcal{H}_m\right] =& \frac{-1}{m}\sum_{\mu=0}^{m-1}\sum_{jkl}([\mathcal{M}^{(3)}, \mathcal{L}^\mu])_{kj} q_j (\mathcal{L}^{m-\mu-1})_{jl} \\ &+ \frac{-1}{m}\sum_{\mu=0}^{m-1}\sum_{jkl}(\mathcal{L}^\mu)_{kj}[q_j, \mathcal{H}_3](\mathcal{L}^{m-\mu-1})_{jl} \\ &+ \frac{-1}{m}\sum_{\mu=0}^{m-1}\sum_{jkl}(\mathcal{L}^\mu)_{kj} q_j([\mathcal{M}^{(3)}, \mathcal{L}^{m-\mu-1}])_{jl}\,.\end{aligned} \tag{2.55}$$

But $\sum_k \mathcal{M}^{(3)}_{jk} = 0$ and $\sum_j \mathcal{M}^{(3)}_{jk} = 0$ (see Exercise 3.14), so this can be written as

$$\begin{aligned}\left[\mathcal{J}_2, \mathcal{H}_m\right] =& \frac{1}{m}\sum_{\mu=0}^{m-1}\sum_{jj'kl}(\mathcal{L}^\mu)_{kj}\Big((\mathcal{M}^{(3)})_{jj'} q_{j'} - q_j(\mathcal{M}^{(3)})_{jj'}\Big)(\mathcal{L}^{m-\mu-1})_{j'l} \\ &+ \frac{-1}{m}\sum_{\mu=0}^{m-1}\sum_{jkl}(\mathcal{L}^\mu)_{kj}[q_j, \mathcal{H}_3](\mathcal{L}^{m-\mu-1})_{jl}\,.\end{aligned} \tag{2.56}$$

To simplify these expressions, let us note the identity

$$\mathcal{M}^{(3)}_{jk} q_k - q_j \mathcal{M}^{(3)}_{jk} = (\mathcal{L}^2)_{jk} + \delta_{jk}[q_j, \mathcal{H}_3]\,, \tag{2.57}$$

which is left as Exercise 3.15. We get

$$\left[\mathcal{J}_2, \mathcal{H}_m\right] = \frac{1}{m}\sum_{\mu=0}^{m-1}\sum_{kl}(\mathcal{L}^{m+1})_{kl} = (m+1)\mathcal{H}_{m+1}\,, \tag{2.58}$$

as we claimed.

3 Other remarks

3.1 Brief bibliography

A physical motivation for quantum mechanics and Schrödinger's equation can be found in [9, § 16.5]. A mathematical point of view can be found in [13, 14]. An introduction to quantum integrability can be found in [8, § 5.1-5.2] and [12, § 4.1]. We refer to [7] for a discussion on different notions of quantum integrability.

The harmonic oscillator is a particularly nice toy model described in many books. Let us mention [1, Example 4.33] for a construction of the action-angle coordinates of the classical system, and [14, § 2.6] for a rigorous derivation of the eigenfunctions of the quantum system.

The Calogero-Moser system was originally introduced as a quantum system by Calogero [6]. Its classical integrability was proved by Moser [11] who exhibited the Lax pair that we used. The change of coordinates that allowed us to prove Liouville integrability of the system in §2.3.1 was observed by Kazhdan, Kostant and Sternberg [10]. The particular r-matrices that we have used in §2.3.2 come from [3, 5]. For more on the use of r-matrices for integrable systems, we refer to [2, 4].

For the integrability of the quantum Calogero-Moser system, we have followed the argument of Ujino, Hikami and Wadati [15, 16]. We refer to [2, § 3.3-3.4] for more details on the use of quantum Lax pairs and R-matrices for Calogero-Moser systems, as well as for the construction of their eigenfunctions. A different method to prove the quantum integrability of the system consists in using Dunkl operators, see [8, § 6].

3.2 Exercises

Exercise 3.1. *Check the dependence relation in (1.15) by proving that it vanishes when applied to a function. Can you find other polynomials P such that $P(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = 0$?*

Exercise 3.2. *Check that the functions (h_i) given in (2.2) are Poisson commuting under the canonical Poisson bracket $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = 0 = \{p_i, p_j\}$.*
Furthermore, check that the corresponding operators $(\mathcal{H}_i)$ given in (2.6) are pairwise commuting.

Exercise 3.3. *In this exercise, we formally build eigenfunctions and eigenvalues[6] for the operator $\mathcal{H} = \frac{1}{2}\frac{d^2}{dq^2} + \omega^2 q^2$.*

a) Assume that the function

$$\phi(q) = \exp\left(\frac{\mathrm{i}\omega q^2}{\sqrt{2}}\right) P(q)$$

is an eigenfunction of $\mathcal{H}$ for some polynomial $P(q)$ and eigenvalue E. Show that the polynomial $P(q)$ satisfies

$$\frac{d^2y}{dq^2} + 2\sqrt{2}\mathrm{i}\omega q \frac{dy}{dq} + 2\left(\frac{\mathrm{i}\omega}{\sqrt{2}} - E\right) y = 0\,.$$

b) Show that for any $n \geq 0$, the Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

satisfies Hermite's equation $y'' - 2xy' + 2\lambda y = 0$ with $\lambda = n$.

c) Deduce that the following pairs are eigenfunctions and eigenvalues of $\mathcal{H}$ for each $n \geq 0$

$$\phi_n = \exp\left(\frac{\mathrm{i}\omega q^2}{\sqrt{2}}\right) H_n\left(\sqrt{-\mathrm{i}\sqrt{2}\omega}\; q\right)\,, \quad E_n = \mathrm{i}\sqrt{2}\,\omega\left(n + \frac{1}{2}\right)\,.$$

Exercise 3.4. *Prove that the classical Lax equation (2.12) for the Calogero-Moser system is satisfied.*

[6] The eigenfunctions and eigenvalues that we obtain differ from those in most textbooks due to our choice of $\hbar$ made in Remark 2.1. Reintroducing $\hbar$ and taking it to be positive, we get real eigenfunctions with real eigenvalues for the corresponding self-adjoint operator.

Exercise 3.5. *Verify that the Calogero-Moser Hamiltonian* H_{CM} (2.14) *is* $\frac{1}{2}\operatorname{tr} L^2$.

Exercise 3.6. *Verify that the pairs* (Q, L) *and* $(\tilde{Q}, \tilde{L})$ *satisfy* (2.16).

Exercise 3.7. *Let* $U \in \mathrm{U}(n)$ *be a matrix diagonalising* L (2.11a). *By conjugating* (2.16) *with* U *and looking at the diagonal entries of this identity, show that* Uv *can not have an entry equal to zero. Deduce that we can find such a* U *satisfying* $Uv = v$.

Exercise 3.8. *Show that the Lax matrix* L *from* (2.11a) *and the element* r_{12} *from* (2.40) *are such that they satisfy* (2.36) *if the space is endowed with the Poisson bracket* $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = 0 = \{p_i, p_j\}$.
Hint: *Prove the identity at each entry* (ij, kl) *by showing that*

$$\begin{aligned}
\{L_1 \overset{\otimes}{,} L_2\}_{ij,kl} =& - \mathrm{i}g(\delta_{il} - \delta_{kj}) \left[\frac{\delta_{ij}\delta_{(k\neq l)}}{(q_k - q_l)^2} + \frac{\delta_{kl}\delta_{(i\neq j)}}{(q_i - q_j)^2} \right] , \\
[a_{12}, L_1]_{ij,kl} =& \mathrm{i}g \frac{\delta_{il}\delta_{(k\neq l)}\delta_{(k\neq j)}}{(q_j - q_k)(q_k - q_l)} - \mathrm{i}g \frac{\delta_{kj}\delta_{(k\neq l)}\delta_{(i\neq l)}}{(q_i - q_l)(q_l - q_k)} , \\
-[a_{21}, L_2]_{ij,kl} =& \mathrm{i}g \frac{\delta_{il}\delta_{(i\neq j)}\delta_{(k\neq j)}}{(q_k - q_j)(q_j - q_i)} - \mathrm{i}g \frac{\delta_{kj}\delta_{(i\neq j)}\delta_{(i\neq l)}}{(q_l - q_i)(q_i - q_j)} , \\
[b_{12}, L_1]_{ij,kl} - [b_{21}, L_2]_{ij,kl} =& \mathrm{i}g \frac{\delta_{(i\neq j)}\delta_{(k\neq l)}}{(q_i - q_j)(q_k - q_l)} (\delta_{il} - \delta_{jk}) ,
\end{aligned}$$

where $\delta_{(j\neq k)} = 1 - \delta_{jk}$ *equals* $+1$ *if* $j \neq k$ *and* 0 *when* $j = k$.

Exercise 3.9. *Similarly to the previous exercise, show that the Lax matrix* L *from* (2.11a) *and the element* $\tilde{r}_{12}$ *given by*

$$\tilde{r}_{12} = \sum_{i\neq j} \frac{1}{q_i - q_j} E_{ij} \otimes E_{ji} + \frac{1}{2} \sum_{i\neq j} \frac{1}{q_i - q_j} E_{ii} \otimes (E_{ij} - E_{ji}) ,$$

are such that they satisfy (2.36). *Deduce that whenever you are given a pair* (L, r) *satisfying* (2.36), *then* r *is not uniquely determined.*

Exercise 3.10. *Check that Jacobi identity* (2.42) *is equivalent to* (2.43) *under* (2.36).

Exercise 3.11. *Prove that the quantum Lax equation* (2.19) *for the Calogero-Moser system is satisfied.*
Hint: *the diagonal entry* (k, k) *and the off-diagonal entry* (j, k) *are respectively given by*

$$-2 \sum_{s\neq k} \frac{a^2 - \mathrm{i}a}{(q_s - q_k)^3} , \quad \textit{and} \quad \frac{\mathrm{i}a}{(q_j - q_k)^2} \left(\frac{2}{q_j - q_k} + \frac{\partial}{\partial q_k} - \frac{\partial}{\partial q_j} \right) .$$

Exercise 3.12. *Verify that the quantum Calogero-Moser Hamiltonian* (2.17) *coincides with* $\mathcal{H}_2$ *defined in* (2.25).
Hint: *show the identity*

$$\sum_{1\leq j\leq n} \sum_{\substack{1\leq k\leq n \\ k\neq j}} \sum_{\substack{1\leq s\leq n \\ s\neq j,k}} \frac{1}{(q_s - q_j)(q_s - q_k)} = 0 ,$$

by rewriting the sums over indices $1 \leq a < b < c \leq n$.

Exercise 3.13. *We prove that* $[\mathcal{H}_1, \mathcal{H}_m] = 0$ *for all* $m \geq 1$.

a) *Show that* $\mathcal{H}_1 = \sum_j \partial/\partial q_j$.

b) *Derive* (2.46)*, and deduce the desired result.*

Exercise 3.14. *We prove that* $[\mathcal{H}_3, \mathcal{H}_m] = 0$ *for all* $m \geq 1$.

a) *Show that the third quantum Calogero-Moser Hamiltonian* $\mathcal{H}_3$ *is given by* (2.47).

b) *Derive the quantum Lax equation* (2.48a).

Hint: *the diagonal entry* (j,j) *and the off-diagonal entry* (j,k) *are respectively given by*

$$\underline{(j,j):} \quad 2g^2 \sum_{r\neq j} \frac{1}{(q_j - q_r)^3} \left(\frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_r} \right), \qquad \text{for } g^2 = a^2 - \mathrm{i}a\,,$$

$$\underline{(j,k):} \quad \mathrm{i}a \left(\frac{2}{(q_j - q_k)^3} \left(\frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_k} \right) - \frac{1}{(q_j - q_k)^2} \left(\frac{\partial^2}{\partial q_j^2} - \frac{\partial^2}{\partial q_k^2} \right) \right) + \mathrm{i}a\, g^2 \frac{1}{(q_j - q_k)^2} \sum_{r \neq j,k} \left(\frac{1}{(q_k - q_r)^2} - \frac{1}{(q_j - q_r)^2} \right).$$

c) *Check that the matrix* $\mathcal{M}^{(3)}$ (2.48b) *satisfies* $\sum_j \mathcal{M}^{(3)}_{jk} = 0 = \sum_k \mathcal{M}^{(3)}_{jk}$.

d) *Deduce the desired result.*

Exercise 3.15. *Prove the identity* (2.57)*, where* $\mathcal{M}^{(3)}$ *is given by* (2.48b) *and* $\mathcal{L}$ *by* (2.18a)*.*
Hint: *the diagonal entry* (j,j) *and the off-diagonal entry* (j,k) *are respectively given by*

$$\underline{(j,j):} \quad \mathrm{i}a \sum_{l\neq j} \frac{1}{(q_j - q_l)^2}\,,$$

$$\underline{(j,k):} \quad \frac{\mathrm{i}a}{(q_j - q_k)} \left(\frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_k} \right) - \frac{\mathrm{i}a}{(q_j - q_k)^2} - a^2 \sum_{l \neq j,k} \frac{1}{(q_j - q_l)(q_l - q_k)}\,.$$

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