Multiplicative quiver varieties and generalised Ruijsenaars-Schneider models

OLEG CHALYKH & MAXIME FAIRON

UNIVERSITY OF LEEDS

INTRODUCTION

The Calogero-Moser system can be obtained by performing a Hamiltonian reduction on the cotangent bundle for $\mathfrak{gl}_n(\mathbb{C})$. One can also view this reduction as performed on the cotangent bundle for the space of representations of a one-loop quiver. Our objective is to adapt this result to the Ruijsenaars-Schneider system and its variants, while transferring as much information as possible to the algebra defined by the quiver.

METHOD

Van den Bergh's work [3] is about translating a quasi-Hamiltonian reduction

$$\mathcal{R}ep \twoheadrightarrow \mathcal{R}ep//G \qquad (1)$$

at the algebra level, for a Lie group action $G \curvearrowright \mathcal{R}ep$ on the representation space of an algebra. There is an **explicit construction** in the case of the **multiplicative preprojective algebra** of an **arbitrary quiver**.

To use this formalism, we need to define on a noncommutative algebra A :

- A double quasi-Poisson bracket $\{\!\{-,-\}\!\} : A^{\otimes 2} \to A^{\otimes 2}$
- A multiplicative moment map $\Phi \in A$ compatible with the double bracket

Then, we consider :

- A dimension vector α
- An element g_q in a Lie group $G(\alpha)$

We can form the representation space $\mathcal{R}ep(A,\alpha)$, and any element $a \in A$ becomes represented by a matrix $\mathcal{X}(a) \in \mathcal{O}(\mathcal{R}ep(A,\alpha))$. Moreover, $\mathcal{X}(\Phi)$ is $G(\alpha)$-valued.

For suitable choices, $\mathcal{R}ep(A,\alpha)$ is endowed with a quasi-Hamiltonian structure and the GIT quotient

$$\mathcal{R}ep(A,\alpha)///_q\, G(\alpha) \;=\; \mathcal{X}(\Phi)^{-1}(g_q)//G(\alpha)$$

is a smooth variety with a nondegenerate Poisson bracket $\{-,-\}_{\mathrm{P}}$.

The Poisson bracket $\{-,-\}_{\mathrm{P}}$ is solely defined by the double bracket $\{\!\{-,-\}\!\}$. If we write $m : A^{\otimes 2} \to A$ the multiplication on A, we get that $\{-,-\} := m \circ \{\!\{-,-\}\!\}$ is a Loday bracket, such that for any $a,b \in A$,

$$\{\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)\}_{\mathrm{P}} = \mathrm{tr}\,\mathcal{X}\left(\{a,b\}\right) \qquad (2)$$

Note that any function on the reduced space is of the particular form $\mathrm{tr}\,\mathcal{X}(a) \in \mathcal{O}(\mathcal{R}ep(A,\alpha))^{G(\alpha)}$.

QUIVER CASE

Consider a quiver Q and let $\bar{Q}$ be its double. Van den Bergh defined a **double quasi-Hamiltonian structure** on $A = \mathbb{C}\bar{Q}_{(1+aa^*)_{a\in\bar{Q}}}$ (the localisation of the path algebra at $(1+aa^*)_{a\in\bar{Q}}$). We set $\epsilon : \bar{Q} \to \{\pm 1\}$ to be $+1$ on elements of Q, the initial quiver, and to be -1 on $\bar{Q} - Q$, the set of double arrows.

Proposition 1 [2, Section 2]. *For a suitable ordering $<$ on $\bar{Q}$, Van den Bergh's double bracket is given by*

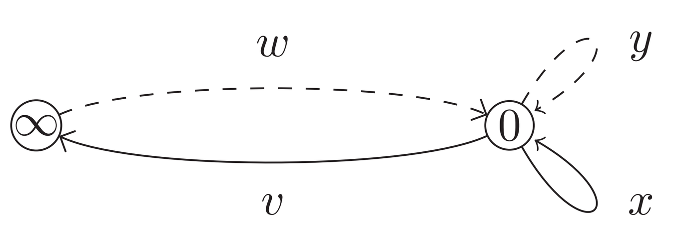
$$\begin{aligned}
\{\!\{a,a\}\!\} &= \frac{1}{2}\epsilon(a)\left(a^2 \otimes e_{t(a)} - e_{h(a)} \otimes a^2\right) && (a \in \bar{Q})\\
\{\!\{a,a^*\}\!\} &= e_{h(a)} \otimes e_{t(a)} + \frac{1}{2}a^*a \otimes e_{t(a)} + \frac{1}{2}e_{h(a)} \otimes aa^*\\
&\quad + \frac{1}{2}(a^* \otimes a - a \otimes a^*)\delta_{h(a),t(a)} && (a \in Q)\\
\{\!\{a,b\}\!\} &= \frac{1}{2}e_{h(a)} \otimes ab + \frac{1}{2}ba \otimes e_{t(a)} && (a<b,\ b \neq a^*)\\
&\quad - \frac{1}{2}(b \otimes a)\delta_{h(a),h(b)} - \frac{1}{2}(a \otimes b)\delta_{t(a),t(b)},
\end{aligned}$$

where $t(a)$ (resp. $h(a)$) denotes the vertex which is the tail (resp. head) of a, and the (e_i) are elements of the algebra indexed by the vertex set such that $e_ie_j = \delta_{ij}e_i$.

The multiplicative moment map Φ corresponding to this structure comes from the defining relation for the multiplicative preprojective algebra associated to Q.

ONE-LOOP QUIVER

We look at the quiver $\bar{Q}$, which is the double of the one-loop quiver x with an extra arrow.



Set A for the localisation of the path algebra $\mathbb{C}\bar{Q}$ at $(1+aa^*)_{a\in\bar{Q}}$. We have for all $k,l \geq 0$:

$$\{x^k,x^l\} = 0, \quad \{y^k,y^l\} = 0,$$
$$\{(xy)^k,(xy)^l\} = 0, \quad \{z^k,z^l\} = 0,$$

where $\{-,-\}$ is the left Loday bracket obtained from Van den Bergh's double bracket, and the element $z = y + x^{-1}$ is defined after localising A at x.

A suitable moduli space of representations is given by the set of matrices

$$X, Y \in \mathrm{Mat}_{n\times n}(\mathbb{C}), \quad V \in \mathrm{Mat}_{n\times 1}(\mathbb{C}), \quad W \in \mathrm{Mat}_{1\times n}(\mathbb{C}),$$

satisfying $(\mathrm{Id}_n + XY)(\mathrm{Id}_n + YX)^{-1}(\mathrm{Id}_n + VW) = q_0\,\mathrm{Id}_n$ for some $q_0 \in \mathbb{C}^\times$, under the identification of the orbits of the action of $\mathrm{GL}_n(\mathbb{C})$ as $g.(X,Y,V,W) = (gXg^{-1}, gYg^{-1}, gV, Wg^{-1})$, for $g \in \mathrm{GL}_n(\mathbb{C})$. We write this space $\mathcal{C}_{n,q_0}$.
On a dense open subset $\mathcal{C}^\circ_{n,q_0}$ of $\mathcal{C}_{n,q_0}$, each orbit contains an element (X,Y,V,W) such that, for a set of log-canonical coordinates (x_i,σ_i), we have $X = A$ with $A = \mathrm{diag}(x_1,\dots,x_n)$ and

$$Y + X^{-1} = (L_{ij}) \text{ for } L_{ij} = \frac{(q_0-1)\sqrt{\sigma_i\sigma_j}}{q_0 - x_ix_j^{-1}} \prod_{l\neq i}\sqrt{\frac{q_0 - x_lx_i^{-1}}{1 - x_lx_i^{-1}}} \prod_{k\neq j}\sqrt{\frac{q_0 - x_kx_j^{-1}}{1 - x_kx_j^{-1}}} \qquad (3)$$

This is the **Lax matrix** for the **trigonometric Ruijsenaars-Schneider model**.

The elements in involution at the algebra level give that the following families can define integrable systems :

$$\{\mathrm{tr}\,A^k\}_{k\in\mathbb{N}}, \quad \{\mathrm{tr}(L-A)^k\}_{k\in\mathbb{N}}, \quad \{\mathrm{tr}(LA)^k\}_{k\in\mathbb{N}}, \quad \{\mathrm{tr}\,L^k\}_{k\in\mathbb{N}}. \qquad (4)$$

Thus, **the double quasi-Poisson structure on A implies the integrability** of the Ruijsenaars-Schneider model (and its variants). See [2, Section 3] for further explanations and discussions of these Hamiltonians.

CYCLIC QUIVER

In the extended one-loop quiver $\bar{Q}$, we can replace the loop x (and its double y) by a cycle consisting of $m \geq 2$ vertices labelled by $\mathbb{Z}_m$ with arrows $x_s : s \to s+1$ (and their doubles $y_s : s+1 \to s$) for all $s \in \mathbb{Z}_m$. By looking at the representation space, we get a smooth Poisson variety $\mathcal{C}_{n,\mathbf{t}}(m)$ after quasi-Hamiltonian reduction, for a regular $\mathbf{t} = (t_s) \in (\mathbb{C}^\times)^m$. We can define a dense open subset $\mathcal{C}^\circ_{n,\mathbf{t}}(m) \subset \mathcal{C}_{n,\mathbf{t}}(m)$ and a map $\xi : \mathcal{C}^\circ_{n,\mathbf{t}}(m) \to \mathcal{C}^\circ_{n,q}$ (the latter is given above by setting $q = t_{m-1}$) such that ξ is an isomorphism of Poisson varieties.

By adapting the argument from the one-loop quiver case, we find **generalised Ruijsenaars-Schneider models** with $(\mathbb{Z}_m)^n$-symmetry. Furthermore, the involution of the different families can be checked at the algebra level. They are defined as

$$\{G_{m,k} := \mathrm{tr}(A^{-1}L^m)^k\}_{k\in\mathbb{N}}, \quad \text{and}$$
$$\{H_{m,k} := \mathrm{tr}(A^{-1}P(L))^k\}_{k\in\mathbb{N}}, \quad \text{for } P(L) = \prod_{s=0}^{m-1}(L - t_s^{-1}\,\mathrm{Id}_n). \qquad (5)$$

Using the map ξ, we can pull back the log-canonical coordinates (x_i,σ_i) at regular points. For any $m \geq 2$, the first element $G_{m,1}$ takes the form

$$G_{m,1} = \sum_{1\leq j_0,\dots,j_{m-1}\leq n} (\sigma_{j_0}\dots\sigma_{j_{m-1}})x_{j_0}^{-1} \prod_{s=0}^{m-1}\frac{t-1}{t - x_{j_s}x_{j_{s+1}}^{-1}} \prod_{s=0}^{m-1}\prod_{a\neq j_s}^{n}\frac{t - x_ax_{j_s}^{-1}}{1 - x_ax_{j_s}^{-1}}. \qquad (6)$$

Note that this expression also makes sense for $m = 0,1$. The second family is more general as in the limit $t_s \to \infty$, the function $H_{m,j}$ tends to $G_{j,m} = \mathrm{tr}(A^{-1}L^m)^j$. It is possible to write explicitly each element $H_{m,1}$ as a linear combination of $\{G_{m',j}\,|\,m' \leq m\}$. For example with $m = 2$

$$H_{2,1} = G_{2,1} - (t_0^{-1} + t_1^{-1})G_{1,1} + (t_0t_1)^{-1}G_{0,1}. \qquad (7)$$

These functions are defined **on the same phase space** as the Hamiltonian for the trigonometric Ruijsenaars-Schneider. Their quantum versions appeared recently in the context of supersymmetric gauge theory and cyclotomic DAHAs, as well as in the context of the Macdonald theory. See [2, Sections 4-5] for additional details.

WORK IN PROGRESS : THE SPIN TRIGONOMETRIC RS MODEL

We are applying this method to obtain the **spin version** of the trigonometric Ruijsenaars-Schneider model, and we can prove the conjecture given in [1] that the space variables (q_i) and spin variables (f_{ij}) form a Poisson algebra with brackets $\{q_i,q_k\} = 0$, $\{f_{ij},q_k\} = -\delta_{jk}f_{ij}$ and

$$\begin{aligned}
\{f_{ij},f_{kl}\} = &\ [\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}) + \coth(q_{li})]f_{ij}f_{kl}\\
&+ [\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}+\gamma) - \coth(q_{il}+\gamma)]f_{il}f_{kj}\\
&+ [\coth(q_{ki}) + \coth(q_{il}+\gamma)]f_{ij}f_{il} + [\coth(q_{jk}) - \coth(q_{jl}+\gamma)]f_{ij}f_{jl}\\
&+ [\coth(q_{ki}) + \coth(q_{kj}+\gamma)]f_{kj}f_{kl} + [\coth(q_{il}) - \coth(q_{lj}+\gamma)]f_{lj}f_{kl}
\end{aligned}$$

where we set $q_{ij} = q_i - q_j$. In fact, the elements (f_{ij}) can be defined from another set of spin variables $(\mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$ by $f_{ij} = \sum_{\alpha=1}^d \mathbf{a}_i^\alpha\mathbf{c}_j^\alpha$, where d is the number of spins. Our formalism allows us to compute the Poisson algebra for the elements $(x_i, \mathbf{a}_i^\alpha, \mathbf{c}_i^\alpha)$.
We are also studying some variants, as well as **generalised spin models** with $(\mathbb{Z}_m)^n$-symmetry.

REFERENCES

[1] G. Arutyunov and S. Frolov. On the Hamiltonian structure of the spin Ruijsenaars-Schneider model. *J. Phys. A*, 31(18):4203–4216, 1998.
[2] O. Chalykh and M. Fairon. Multiplicative quiver varieties and generalised Ruijsenaars-Schneider models. *arXiv:1704.0581 [math.QA]*.
[3] M. Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360(11):5711–5769, 2008.

CONTACT INFORMATION

Maxime Fairon, School of Mathematics,
University of Leeds, LS2 9JT Leeds (UK)
Email : mmmfai@leeds.ac.uk

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