Double multiplicative Poisson vertex algebras

(Algèbres vertex de Poisson multiplicatives doubles)

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Motivation (1)

Example

Volterra lattice eq. on $V = \Bbbk[u_n \mid n \in \mathbb{Z}]$: $(char(\Bbbk) = 0)$

$$\frac{du_n}{dt} = u_n u_{n+1} - u_{n-1} u_n \,, \quad n \in \mathbb{Z}$$

Underlying structure :

– eq. commutes with automorphism $S : u_n \mapsto u_{n+1}$

– Poisson bracket $\{u_m, u_n\} = (\delta_{m,n+1} - \delta_{m,n-1}) u_m u_n$ compatible with S

Intuitively :

– Hamiltonian "$h = \sum_{m \in \mathbb{Z}} u_m = (\sum_{m \in \mathbb{Z}} S^m) u_0$" (! ! !)

Motivation (2) – local lattice PA

Fix V a *commutative* algebra with Poisson bracket

($\hookrightarrow$ bilinear skewsymmetric map $\{-,-\}: V \times V \to V$ + Leibniz rules + Jacobi identity)

Definition

$(V, \{-,-\})$ is a lattice Poisson algebra if it admits an automorphism S of infinite order compatible with $\{-,-\}$, that is $\{S(a), S(b)\} = S(\{a,b\})$.

Furthermore, it is local if, given $a, b \in V$, we have $\{S^n(a), b\} = 0$ for all but finitely many $n \in \mathbb{Z}$.

$\rightsquigarrow$ can define the Laurent polynomial $\{a_\lambda b\} := \sum_{n\in\mathbb{Z}} \{S^n(a), b\}\lambda^n$.

recover $\{a, b\} := \mathrm{mRes}_\lambda \{a_\lambda b\}$ by picking order λ^0

Motivation (3) – equivalence

Fix V a *commutative* algebra with infinite order $S \in \mathrm{Aut}(V)$

Theorem ([De Sole-Kac-Valeri-Wakimoto,'19])

There is a $1-1$ correspondence between the following structures on V :

- *local lattice Poisson algebra (with $\{-,-\}$);*
- *multiplicative Poisson vertex algebra (with $\{-_\lambda -\}$);*

which is given by

$$\{-,-\} \longrightarrow \{a_\lambda b\} := \textstyle\sum_{n\in\mathbb{Z}} \{S^n(a), b\}\lambda^n$$

$$\{a,b\} := \mathrm{mRes}_\lambda \{a_\lambda b\} \longleftarrow \{-_\lambda -\}$$

The second type of structure is obtained by translating properties :

compatibility with $S \leftrightarrow$ sesquilinearity

skewsymmetry $\leftrightarrow$ "skewsymmetry"

Leibniz rules $\leftrightarrow$ left/right Leibniz rules

Jacobi identity $\leftrightarrow$ "Jacobi identity"

Example (Volterra lattice)

$\{u_m, u_n\} = (\delta_{m,n+1} - \delta_{m,n-1})u_m u_n$ is equivalent to

$\{u_\lambda u\} = u\,\lambda S(u) - u\,\lambda^{-1} S^{-1}(u)$ for $u := u_0$

Motivation (4) – MPVA

Fix V a *commutative* algebra with infinite order $S \in \mathrm{Aut}(V)$

Definition ([De Sole-Kac-Valeri-Wakimoto,'19,'20])

A multiplicative λ-bracket on V is a linear map

$$\{-_\lambda-\}: V \otimes V \to V[\lambda^{\pm 1}], \quad a \otimes b \mapsto \{a_\lambda b\}\,, \text{ such that}$$

$$\{S(a)_\lambda b\} = \lambda^{-1}\{a_\lambda b\}\,, \quad \{a_\lambda S(b)\} = \lambda S(\{a_\lambda b\})\,, \qquad \text{(sesquilinearity)}$$

$$\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}\,, \qquad \text{(left Leibniz rule)}$$

$$\{ab_\lambda c\} = \{a_{\lambda x} c\}\left(\big|_{x=S} b\right) + \left(\big|_{x=S} a\right)\{b_{\lambda x} c\}\,. \qquad \text{(right Leibniz rule)}$$

V is a multiplicative Poisson vertex algebra if moreover

$$\{a_\lambda b\} = -\big|_{x=S}\{b_{\lambda^{-1}x^{-1}} a\}\,, \qquad \text{(skewsymmetry)}$$

$$\{a_\lambda\{b_\mu c\}\} - \{b_\mu\{a_\lambda c\}\} - \{\{a_\lambda b\}_{\lambda\mu} c\} = 0\,. \qquad \text{(Jacobi identity)}$$

Motivation (5) – DΔE

Fix V a multiplicative Poisson vertex algebra (for S, $\{-_\lambda-\}$)
Let $\overline{V} := V/(S-1)V$, with elements denoted $\int f$ ("*local functionals*")

Proposition ([De Sole-Kac-Valeri-Wakimoto,'19,'20])

We have that $\overline{V}$ is a Lie algebra for $\{\int f, \int g\} := \int \{f_\lambda g\}\big|_{\lambda=1}$

Furthermore, $\overline{V}$ acts by derivations on V through $\{\int f, g\} := \{f_\lambda g\}\big|_{\lambda=1}$ and such derivations commute with S.

Example (Volterra lattice)

Recall $\{u_\lambda u\} = u\,\lambda S(u) - u\,\lambda^{-1}S^{-1}(u)$ for $u := u_0$ on $V = \Bbbk[u_n \mid n \in \mathbb{Z}]$

Then $\int u$ is such that

$$\frac{du_n}{dt} := \{\textstyle\int u, u_n\} = \Big((\lambda S)^n \{u_\lambda u\}\Big)\Big|_{\lambda=1} = u_n u_{n+1} - u_n u_{n-1}$$

(so $\int u$ allows to make sense of "$h = \sum_m S^m(u)$")

Plan for the talk

1. **Definition and properties of DMPVA**
2. Application to integrable systems
3. Non-local and rational cases

Double Poisson brackets (1)

$\mathcal{V}$ denotes an associative unital algebra over $\Bbbk$

For $d \in \mathcal{V}^{\otimes 2}$, set $d = d' \otimes d''(= \sum_k d'_k \otimes d''_k)$, and $d^\sigma = d'' \otimes d'$ $(\otimes = \otimes_{\Bbbk})$
Multiplication on $\mathcal{V}^{\otimes 2}$: $(a \otimes b)(c \otimes d) = ac \otimes bd$.

Definition ([Van den Bergh, *double Poisson algebras*,'08])

A double bracket on $\mathcal{V}$ is a $\Bbbk$-linear map $\{\!\{-,-\}\!\} : \mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\ \{\!\{b,a\}\!\}^\sigma$ ("cyclic" skewsymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1)\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}(1 \otimes c)$ (left Leibniz rule)
3. $\{\!\{ac,b\}\!\} = (1 \otimes a)\{\!\{c,b\}\!\} + \{\!\{a,b\}\!\}(c \otimes 1)$ (right Leibniz rule)

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To shorten notations, use the $\mathcal{V}$-bimodule structures on $\mathcal{V}^{\otimes 2}$

$$a(d' \otimes d'')b := ad' \otimes d''b\,, \qquad a * (d' \otimes d'') * b := d'b \otimes ad''$$

$$\Rightarrow \{\!\{a,bc\}\!\} = b\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}c\,, \qquad \{\!\{ac,b\}\!\} = a * \{\!\{c,b\}\!\} + \{\!\{a,b\}\!\} * c$$

Double Poisson brackets (2)

Let $\mathcal{V}$ be equipped with a double bracket $\{\!\{-,-\}\!\}$

Definition ([Van den Bergh,'08], [De Sole-Kac-Valeri,'15])

$(\mathcal{V}, \{\!\{-,-\}\!\})$ is a double Poisson algebra if $\forall a,b,c \in \mathcal{V}$

$$\{\!\{a, \{\!\{b,c\}\!\}\}\!\}_L - \{\!\{b, \{\!\{a,c\}\!\}\}\!\}_R - \{\!\{\{\!\{a,b\}\!\}, c\}\!\}_L = 0\,. \quad \text{(double Jacobi identity)}$$

for $\{\!\{a, b' \otimes b''\}\!\}_L = \{\!\{a, b'\}\!\} \otimes b''$, $\{\!\{a, b' \otimes b''\}\!\}_R = b' \otimes \{\!\{a, b''\}\!\}$,
$\{\!\{a' \otimes a'', b\}\!\}_L = \{\!\{a', b\}\!\} \otimes_1 a'' := \{\!\{a', b\}\!\}' \otimes a'' \otimes \{\!\{a', b\}\!\}''$.

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$\{\!\{a' \otimes a'', b\}\!\}_L = \{\!\{a',b\}\!\} \otimes_1 a'' := \{\!\{a',b\}\!\}' \otimes a'' \otimes \{\!\{a',b\}\!\}''$.

On rep. space : $a \in \mathcal{V} \rightsquigarrow$ "matrix entry" $a_{ij} \in \mathcal{V}_N := \mathbb{C}[\mathrm{Rep}(\mathcal{V},N)]$

Theorem ([Van den Bergh,'08])

If $(\mathcal{V}, \{\!\{-,-\}\!\})$ is a double Poisson algebra, then $\mathcal{V}_N$ has a unique Poisson bracket $\{-,-\}$ satisfying

$$\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}\,.$$

(squared) NC Volterra lattice

Example

$\mathcal{V} = \Bbbk\langle u_n \mid n \in \mathbb{Z}\rangle$ has double Poisson bracket

$$\{\!\{u_m, u_n\}\!\} = (\delta_{m,n+1} - \delta_{m,n-1}) u_n u_m \otimes u_m u_n$$

Thm. $\Rightarrow \mathcal{V}_N = \Bbbk[u_{n,ij} \mid n \in \mathbb{Z}, 1 \le i,j \le N]$, $N \geq 1$, has a Poisson bracket

$$\{u_{m,ij}, u_{n,kl}\} = (\delta_{m,n+1} - \delta_{m,n-1}) (u_n u_m)_{kj} \, (u_m u_n)_{il}$$

For $N = 1$, $V = \Bbbk[\hat{u}_n := u_{n,11} \mid n \in \mathbb{Z}] \simeq \mathcal{V}_1$ has Poisson bracket

$$\{\hat{u}_m, \hat{u}_n\} = (\delta_{m,n+1} - \delta_{m,n-1}) \hat{u}_m \hat{u}_n \, \hat{u}_n \hat{u}_m = (\delta_{m,n+1} - \delta_{m,n-1}) \hat{u}_m^2 \hat{u}_n^2$$

(This is the square of the PB for Volterra lattice)

Note : the double Poisson structure is compatible with $S : u_n \mapsto u_{n+1}$
$\rightsquigarrow$ *lattice* double Poisson algebra

Local lattice DPA

(From now on, mainly follow [F.-Valeri,'21 / arXiv :2110.03418])

Definition

$(\mathcal{V}, \{\!\{-,-\}\!\})$ is a lattice double Poisson algebra if it admits an infinite order automorphism $S \in \mathrm{Aut}(\mathcal{V})$ compatible with its double Poisson bracket :

$$\{\!\{S(a), S(b)\}\!\} = S(\{\!\{a,b\}\!\}) := S^{\otimes 2}\,\{\!\{a,b\}\!\}.$$

Furthermore, it is local if, given $a, b \in \mathcal{V}$, we have $\{\!\{S^n(a), b\}\!\} = 0$ for all but finitely many $n \in \mathbb{Z}$.

$\rightsquigarrow$ Laurent polynomial $\{\!\{a_\lambda b\}\!\} := \sum_{n\in\mathbb{Z}} \{\!\{S^n(a), b\}\!\}\,\lambda^n \in \mathcal{V}^{\otimes 2}[\lambda^{\pm 1}]$.

recover $\{\!\{a,b\}\!\} := \mathrm{mRes}_\lambda\,\{\!\{a_\lambda b\}\!\}$ by picking order λ^0

NC equivalence

Fix $\mathcal{V}$ an *associative* algebra with infinite order $S \in \mathrm{Aut}(\mathcal{V})$

Theorem ([F.-Valeri,'19])

There is a $1-1$ correspondence between the following structures on $\mathcal{V}$:

- *local lattice double Poisson algebra (with $\{\!\{-,-\}\!\}$);*
- *double multiplicative Poisson vertex algebra (with $\{\!\{-_\lambda-\}\!\}$);*

which is given by

$$\{\!\{-,-\}\!\} \longrightarrow \{\!\{a_\lambda b\}\!\} := \sum_{n\in\mathbb{Z}} \{\!\{S^n(a),b\}\!\}\,\lambda^n$$

$$\{\!\{a,b\}\!\} := \mathrm{mRes}_\lambda \{\!\{a_\lambda b\}\!\} \longleftarrow \{\!\{-_\lambda-\}\!\}$$

The second type of structure is obtained by translating properties :

compatibility with S ; Leibniz rules ; "cyclic" skewsymmetry ; double Jacobi identity

Example (squared Volterra lattice)

$\{\!\{u_m,u_n\}\!\} = (\delta_{m,n+1}-\delta_{m,n-1})u_n u_m \otimes u_m u_n$ on $\mathcal{V}=\Bbbk\langle u_n \mid n\in\mathbb{Z}\rangle$

$\{\!\{u_\lambda u\}\!\} = \sum_{\epsilon=\pm1} \epsilon\, u S^\epsilon(u) \otimes S^\epsilon(u) u\;\lambda^\epsilon$ for $u := u_0$

double MPVA

Fix $\mathcal{V}$ an *associative* algebra with infinite order $S \in \mathrm{Aut}(\mathcal{V})$

Definition ([F.-Valeri,'21], see also [Casati-Wang,'21])

A double multiplicative λ-bracket on $\mathcal{V}$ is a linear map

$$\{\!\{-_\lambda-\}\!\} : \mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}[\lambda^{\pm 1}], \quad a \otimes b \mapsto \{\!\{a_\lambda b\}\!\}$$, such that

$$\{\!\{S(a)_\lambda b\}\!\} = \lambda^{-1}\{\!\{a_\lambda b\}\!\}, \quad \{\!\{a_\lambda S(b)\}\!\} = \lambda S(\{\!\{a_\lambda b\}\!\}), \qquad \text{(sesquilinearity)}$$

$$\{\!\{a_\lambda bc\}\!\} = \{\!\{a_\lambda b\}\!\} c + b \{\!\{a_\lambda c\}\!\}, \qquad \text{(left Leibniz rule)}$$

$$\{\!\{ab_\lambda c\}\!\} = \{\!\{a_{\lambda x} c\}\!\} *_1 \left(\Big|_{x=S} b\right) + \left(\Big|_{x=S} a\right) *_1 \{\!\{b_{\lambda x} c\}\!\}. \qquad \text{(right Leibniz rule)}$$

$\mathcal{V}$ is a double multiplicative Poisson vertex algebra if moreover

$$\{\!\{a_\lambda b\}\!\} = -\Big|_{x=S} \{\!\{b_{\lambda^{-1}x^{-1}} a\}\!\}^\sigma, \qquad \text{(skewsymmetry)}$$

$$\{\!\{a_\lambda \{\!\{b_\mu c\}\!\}\}\!\}_L - \{\!\{b_\mu \{\!\{a_\lambda c\}\!\}\}\!\}_R - \left\{\!\!\left\{ \{\!\{a_\lambda b\}\!\}_{\lambda\mu}\, c \right\}\!\!\right\}_L = 0. \qquad \text{(Jacobi identity)}$$

for $\{\!\{a_\lambda b' \otimes b''\}\!\}_L = \{\!\{a_\lambda b'\}\!\} \otimes b''$, $\{\!\{a_\lambda b' \otimes b''\}\!\}_R = b' \otimes \{\!\{a_\lambda b''\}\!\}$,
$\{\!\{a' \otimes a''{}_\lambda b\}\!\}_L = \{\!\{a'{}_{\lambda x} b\}\!\} \otimes_1 \left(\Big|_{x=S} a''\right)$.

There is a *master formula* to write $\{\!\{-_\lambda-\}\!\}$ easily, e.g. with generators

Link to representation spaces (1)

Recall $\mathcal{V} \rightsquigarrow \mathcal{V}_N$ parametrised by "matrix entries" a_{ij}, $a \in \mathcal{V}$, $1 \leq i,j \leq N$
Extend $S \in \mathrm{Aut}(\mathcal{V})$ through $S(a_{ij}) = (S(a))_{ij}$

Theorem ([F.-Valeri,'21])

Assume that $\{\!\{-_\lambda-\}\!\}$ is a double multiplicative λ-bracket on $\mathcal{V}$. Then there is a unique multiplicative λ-bracket on $\mathcal{V}_N$ which satisfies

$$\{a_{ij}\,{}_\lambda b_{kl}\} = \sum_{n\in\mathbb{Z}} (a_n b)'_{kj} (a_n b)''_{il} \lambda^n \,,$$

$$\textit{where } \{\!\{a_\lambda b\}\!\} = \sum_{n\in\mathbb{Z}} ((a_n b)' \otimes (a_n b)'')\lambda^n \,.$$

Furthermore, if $(\mathcal{V}, \{\!\{-_\lambda-\}\!\})$ is a double multiplicative Poisson vertex algebra, then $(\mathcal{V}_N, \{-_\lambda-\})$ is a multiplicative Poisson vertex algebra.

Link to representation spaces (2)

Combining this Theorem with the one of Van den Bergh + equivalences :

$$
\begin{array}{ccc}
(\mathcal{V}, \{\!\{-,-\}\!\}, S) & \xleftrightarrow{\text{Equiv of [F-V,'21]}} & (\mathcal{V}, \{\!\{-_\lambda-\}\!\}, S) \\
\Big\downarrow \text{Thm of [VdB,'08]} & & \Big\downarrow \text{Thm of [F-V,'21]} \\
(\mathcal{V}_N, \{-,-\}, S) & \xleftrightarrow{\text{Equiv of [DS-K-V-W,'19]}} & (\mathcal{V}_N, \{-_\lambda-\}, S)
\end{array}
$$

Theorem ([F.-Valeri,'21])

This diagram commutes.

Plan for the talk

1. Definition and properties of DMPVA
2. **Application to integrable systems***
3. Non-local and rational cases

* Initiated in [Casati-Wang,'21] for $\mathcal{V} = \mathbb{R}\langle u_{i,n} \mid i \in I, n \in \mathbb{Z}\rangle$, $S(u_{i,n}) = u_{i,n+1}$
Ideas follow the application of DPVA from [De Sole-Kac-Valeri,'15]

Trace map and associated Lie bracket

Fix $\mathcal{V}$ a double multiplicative Poisson vertex algebra (for S, $\{\!\{-_\lambda-\}\!\}$)

Let $\mathcal{F} := \mathcal{V}/\big((S-1)\mathcal{V} + [\mathcal{V},\mathcal{V}]\big)$, with elements denoted $\int f$

$$f \in \mathcal{V} \quad \mapsto \quad \underbrace{\mathrm{tr}(f) \in \mathcal{V}/[\mathcal{V},\mathcal{V}]}_{\text{trace functions}} \quad \mapsto \quad \underbrace{\textstyle\int f \in \mathcal{F}}_{\text{local functionals}}$$

Proposition ([F.-Valeri,'21])

We have that $\mathcal{F}$ is a Lie algebra for $\{\int f, \int g\} := \int \mathrm{m}\,\{\!\{f_\lambda g\}\!\}\big|_{\lambda=1}$

(extend multiplication $\mathrm{m} : \mathcal{V}^{\otimes 2} \to \mathcal{V}$ as map $\mathrm{m} : \mathcal{V}^{\otimes 2}[\lambda^{\pm 1}] \to \mathcal{V}[\lambda^{\pm 1}]$)

Furthermore, $\mathcal{F}$ acts by derivations on $\mathcal{V}$ through $\{\int f, g\} := \mathrm{m}\,\{\!\{f_\lambda g\}\!\}\big|_{\lambda=1}$ and such derivations commute with S.

$$\Rightarrow \qquad \{\textstyle\int f_1, \{\int f_2, -\}\} - \{\int f_2, \{\int f_1, -\}\} = \{\{\int f_1, \int f_2\}, -\} \quad \text{on } \mathcal{V}$$

NC D∆E

Given $\int f \in \mathcal{F}$, get a *Hamiltonian equation* on $\mathcal{V}$:

$$\frac{du}{dt} := \{\textstyle\int f, u\} = \mathrm{m}\, \{\!\{ f_\lambda u \}\!\} \big|_{\lambda=1} \qquad \forall u \in \mathcal{V}\,.$$

$\Rightarrow$ NC differential-difference equation commuting with S

Example (squared Volterra lattice)

Recall that we have a DMPVA structure on $\mathcal{V} = \Bbbk\langle u_n \rangle$, $S(u_n) = u_{n+1}$,
$\{\!\{ u_\lambda u \}\!\} = \sum_{\epsilon=\pm 1} \epsilon\, u S^\epsilon(u) \otimes S^\epsilon(u) u\ \lambda^\epsilon$ for $u := u_0$

Then $\int u$ is such that

$$\frac{du}{dt} := \{\textstyle\int u, u\} = \sum_{\epsilon=\pm 1} \epsilon\, u\, S^\epsilon(u^2)\, u = u u_1^2 u - u u_{-1}^2 u$$

Towards integrable systems

On DMPVA $(\mathcal{V}, \{\!\{-_\lambda-\}\!\})$: $\{\int f_1, \{\int f_2, -\}\} - \{\int f_2, \{\int f_1, -\}\} = \{\{\int f_1, \int f_2\}, -\}$

When are derivations $X_k := \{\int f_k, -\}$ on $\mathcal{V}$ commuting ?

e.g. $\{\int f_j, \int f_k\} = 0$, $\forall j, k$

$\Rightarrow$ To have compatible DΔEs on $\mathcal{V}$, need to find such local functionals !

First, we need examples of DMPVA to play with

NC polynomials in $\ell = 1$ variable (1)

Fix $\mathcal{V}$ to be $\Bbbk\langle u_n \mid n \in \mathbb{Z}\rangle$, $S(u_n) = u_{n+1}$. Set $u := u_0$

Lemma

Any DMPVA structure with $\{\!\{u_\lambda u\}\!\} \in \mathcal{V}^{\otimes 2}[\lambda^{\pm 1}]$ *must satisfy*
$\{\!\{u_\lambda u\}\!\} = \sum_{k\in\mathbb{Z}}(f_k\lambda^k - S^{-k} f_k^\sigma \lambda^{-k})$, $f_k = f_k(u, u_1, \dots, u_k)$

Proposition ([VdB,'08 – Powell,'16])

Any DMPVA structure with $\{\!\{u_\lambda u\}\!\} \in \mathcal{V}^{\otimes 2}$ *(no λ !) is s.t.*
$\{\!\{u_\lambda u\}\!\} = \alpha(u \otimes 1 - 1 \otimes u) + \beta(u^2 \otimes 1 - 1 \otimes u^2) + \gamma(u^2 \otimes u - u \otimes u^2)$
for $\alpha\gamma - \beta^2 = 0$.

Not interesting for integrability as $\{\int u^k, u\} = \mathrm{m}\,\{\!\{u^k_\lambda u\}\!\}\,|_{\lambda=1} = 0$, $\forall k \geq 1$

NC polynomials in $\ell = 1$ variable (2)

Fix $\mathcal{V}$ to be $\Bbbk\langle u_n \mid n \in \mathbb{Z}\rangle$, $S(u_n) = u_{n+1}$. Set $u := u_0$
Introduce *bullet* product : $(a' \otimes a'') \bullet (b' \otimes b'') = a'b' \otimes b''a''$

Proposition ([F.-Valeri,'21])

Any DMPVA structure with

$$\{\!\{u_\lambda u\}\!\} = g(u) \bullet r(\lambda S) g(u), \qquad r(z) \in \Bbbk[z^{\pm 1}] \text{ s.t. } r(z^{-1}) = -r(z),$$

is such that $g = (\alpha u + \beta) \otimes (\alpha u + \beta)$ *for* $\alpha, \beta \in \Bbbk$.

Proposition ([F.-Valeri,'21], [Casati-Wang,'21])

Any DMPVA structure with

$$\{\!\{u_\lambda u\}\!\} = f\lambda^k - S^{-k}(f)\lambda^{-k}, \qquad f \in \mathcal{V} \otimes \mathcal{V},\ k \geq 1,$$

is such that $f = g \bullet S^k g$ *for* g *as above.*

No easy commuting local functionals to identify...

NC polynomials in $\ell = 2$ variables (1)

Fix $\mathcal{V}$ to be $\Bbbk\langle u_n, v_n \mid n \in \mathbb{Z}\rangle$, $S(u_n) = u_{n+1}, S(v_n) = v_{n+1}$.
Set $u := u_0, v := v_0$.

If e.g. $\{\!\{u_\lambda u\}\!\} = 0$, guaranteed that $\{\int u^k, \int u^l\} = 0$, $\forall k, l \geq 1$

NC polynomials in $\ell = 2$ variables (1)

Fix $\mathcal{V}$ to be $\Bbbk\langle u_n, v_n \mid n \in \mathbb{Z}\rangle$, $S(u_n) = u_{n+1}, S(v_n) = v_{n+1}$.
Set $u := u_0, v := v_0$.

If e.g. $\{\!\{u_\lambda u\}\!\} = 0$, guaranteed that $\{\int u^k, \int u^l\} = 0$, $\forall k, l \geq 1$

Proposition ([F.-Valeri,'21])

Any DMPVA structure on $\mathcal{V}$ of the form

$$\{\!\{u_\lambda u\}\!\} = 0 = \{\!\{v_\lambda v\}\!\}, \quad \{\!\{u_\lambda v\}\!\} = g\lambda^k, \quad g \in \mathcal{V}^{\otimes 2}$$

is given, modulo translation $(u, v) \mapsto (u + \alpha, v + \beta)$, by

(i) $g = a\, 1 \otimes 1$, $a \in \Bbbk$;

(ii) $g = a\, v \otimes v$, $a \in \Bbbk^\times$;

(iii) $g = a\, u_k \otimes u_k$, $a \in \Bbbk^\times$;

(iv) $g = a\, v \otimes v + b\,[v \otimes u_k + u_k \otimes v] + \frac{b^2}{a}\, u_k \otimes u_k$, $a, b \in \Bbbk^\times$;

(v) $g = a\, vu_k \otimes u_kv + b\,[vu_k \otimes 1 + 1 \otimes u_kv] + \frac{b^2}{a}\, 1 \otimes 1$, $a \in \Bbbk^\times$, $b \in \Bbbk$.

NC polynomials in $\ell = 2$ variables (2)

The 5 DMPVA structures ($\{\!\{u_\lambda v\}\!\} = g\lambda^k$) on $\mathcal{V} = \mathbb{k}\langle u_n, v_n \mid n \in \mathbb{Z}\rangle$ give 5 families of compatible DΔEs (for $k \geq 1$ fixed) with $\frac{d}{dt_j} := \frac{1}{j}\{\int u^j, -\}$

Example

$$(i) \quad \frac{dv}{dt_j} = a\, u_k^{j-1}\,, \quad \frac{du}{dt_j} = 0$$

$$(ii) \quad \frac{dv}{dt_j} = a\, v u_k^{j-1} v\,, \quad \frac{du}{dt_j} = 0$$

$$(iii) \quad \frac{dv}{dt_j} = a\, u_k^{j+1}\,, \quad \frac{du}{dt_j} = 0$$

$$(iv) \quad \frac{dv}{dt_j} = a\, v u_k^{j-1} v + b(v u_k^j + u_k^j v) + \frac{b^2}{a} u_k^{j+1}\,, \quad \frac{du}{dt_j} = 0$$

$$(v) \quad \frac{dv}{dt_j} = a\, v u_k^{j+1} v + b(v u_k^j + u_k^j v) + \frac{b^2}{a} u_k^{j-1}\,, \quad \frac{du}{dt_j} = 0$$

NC polynomials in $\ell = 2$ variables (3)

Slight generalisation of cases (iv)-(v) :

$$\frac{dv}{dt_j} = \alpha v u_k^{j-1} v + (v u_k^j + u_k^j v) + \beta u_k^{j+1}\,, \quad \frac{du}{dt_j} = 0\,, \quad j \in \mathbb{Z}_+\,,$$

These are compatible DΔEs.

They come from $(\int u^j)$ with skewsym. double mult. λ-bracket
$\{\!\{u_\lambda u\}\!\} = 0 = \{\!\{v_\lambda v\}\!\}\,, \quad \{\!\{u_\lambda v\}\!\} = (v \otimes u_k + u_k \otimes v + \alpha v \otimes v + \beta u_k \otimes u_k)\lambda^k$
This operation does not satisfy Jacobi identity when $\alpha\beta \neq 1$

$\Rightarrow$ There is a "weaker" version of DMPVA to get compatible DΔEs
(see Subsect. 6.4.3 in [F.-Valeri,'21], also Sect.5 in [Casati-Wang,'21])

Plan for the talk

1. Definition and properties of DMPVA
2. Application to integrable systems
3. **Non-local and rational cases**

Nonlocal DMPVA

Take the definition of DMPVA and use *nonlocal* map

$$\{\!\{-_\lambda-\}\!\} : \mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}[[\lambda^{\pm 1}]], \quad a \otimes b \mapsto \{\!\{a_\lambda b\}\!\}$$

All properties still make sense !

Example

$\mathcal{V} = \Bbbk\langle u_n \mid n \in \mathbb{Z}\rangle$, $S(u_n) = u_{n+1}$. Set $u := u_0$

We have a *nonlocal* DMPVA structure :

$$\{\!\{u_\lambda u\}\!\} := \sum_{n \in \mathbb{Z}} \mathrm{sgn}(n)\,(uu_n \otimes u_n u)\lambda^n$$

Nonlocal DMPVA and rational operators

In [Casati-Wang,'21], different point of view with operators

e.g. $H = \mathrm{r}_u S \mathrm{r}_u - \mathrm{l}_u S \mathrm{l}_u - \frac{1}{2}\mathrm{a}_u \mathrm{c}_u - \frac{1}{2}\mathrm{c}_u \frac{1+S}{1-S}\mathrm{c}_u$

(for the non-commutative Narita-Itoh-Bogoyavlensky hierarchy)

to be interpreted as

$$\begin{aligned}\{\!\{u_\lambda u\}\!\} :=&(1\otimes u)\lambda S \bullet (1\otimes u) - (u\otimes 1)\lambda S \bullet (u\otimes 1)\\ &- \frac{1}{2}(u\otimes 1 + 1\otimes u)\bullet(u\otimes 1 - 1\otimes u)\\ &- \frac{1}{2}(u\otimes 1 - 1\otimes u)\,\frac{1+\lambda S}{1-\lambda S}\bullet(u\otimes 1 - 1\otimes u)\end{aligned}$$

- ✓ all axioms are formally satisfied
- X it does *not* define a nonlocal DMPVA due to Jacobi identity

  using suitable expansion $\frac{1+\lambda S}{1-\lambda S} = \sum_{n>0}[(\lambda S)^n - (\lambda S)^{-n}]$ for skewsymm.

Rational DMPVA

Use (positive) embedding of rational functions as Laurent series :

$$\iota_+ : \mathbb{k}(z) \hookrightarrow \mathbb{k}((z)) = \{\sum_{n \geq -N} a_n z^n \mid a_n \in \mathbb{k}\} \text{ e.g. } \iota_+(\tfrac{1}{1-z}) = \sum_{n \geq 0} z^n$$

Rational operators $\mathcal{Q}(\mathcal{V}) := \{\sum f_1 \iota_+ r_1(S) \bullet \cdots \bullet f_n \iota_+ r_n(S) \bullet f_{n+1} \in (\mathcal{V} \otimes \mathcal{V})((S))\}$

adjoint $A(S) \mapsto A(S)^* = \sum f_{n+1}^\sigma \iota_+ r_n(S^{-1}) \bullet \cdots \bullet f_2^\sigma \iota_+ r_1(S^{-1}) \bullet f_1^\sigma$

e.g. $A(S) = \iota_+ \frac{1}{1-S} = \sum_{n \geq 0} S^n \quad \rightsquigarrow \quad A(S)^* = \iota_+ \frac{-S}{1-S} = -\sum_{n \geq 1} S^n$

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Definition ([F.-Valeri,'21])

A rational double multiplicative λ-bracket on $\mathcal{V}$ is a double multiplicative λ-bracket (i.e. sesquilinearity/Leibniz rules) with $\{\!\{a_\lambda b\}\!\} = A_{ab}(\lambda)$ being the symbol of an element $A_{ab}(S) \in \mathcal{Q}(\mathcal{V})$

$\mathcal{V}$ is a rational double multiplicative Poisson vertex algebra if moreover

$A_{ab}(\lambda) = -A_{ba}(\lambda)^*$ (skewsymmetry) $\quad + $ (Jacobi identity) as before

A classification result

$$H(S) = (1 \otimes u)\iota_+ a(S) \bullet (1 \otimes u) + (1 \otimes u)\iota_+ b(S) \bullet (u \otimes 1) + (u \otimes 1)\iota_+ c(S) \bullet (1 \otimes u) - (u \otimes 1)\iota_+ a(S^{-1}) \bullet (u \otimes 1)$$

Theorem ([F.-Valeri,'21])

The pseudodifference operator $H(S)$ induces a DMPVA structure of rational type on $\mathcal{V} = \Bbbk\langle u_n \mid n \in \mathbb{Z}\rangle$ through $\{\!\{u_\lambda u\}\!\} = H(\lambda)$ (symbol of H) if and only if for some $k \geq 1$ and $p \in \mathbb{Z}$,

$$a(z) = z^p a_1(z^k)\,, \qquad a_1(z) := \alpha \tfrac{1}{1-z}\,,$$

$$b(z) = c(z) = b_1(z^k)\,, \quad b_1(z) := \beta \tfrac{1+z}{1-z}\,,$$

where $\alpha, \beta \in \Bbbk$ are such that $\alpha(2\beta + \alpha) = 0$.

Example (Casati-Wang operator)

Case $k = 1$, $p = 2$, $\alpha = -1$, $\beta = \frac{1}{2}$

$$H(S) = -\mathrm{r}_u \iota_+ \tfrac{S^2}{1-S} \bullet \mathrm{r}_u + \tfrac{1}{2}\mathrm{r}_u \iota_+ \tfrac{1+S}{1-S} \bullet \mathrm{l}_u + \tfrac{1}{2}\mathrm{l}_u \iota_+ \tfrac{1+S}{1-S} \bullet \mathrm{r}_u - \mathrm{l}_u \iota_+ \tfrac{S^{-1}}{1-S} \bullet \mathrm{l}_u$$

$$= (\mathrm{r}_u S \bullet \mathrm{r}_u - \mathrm{l}_u S^{-1} \bullet \mathrm{l}_u) - \tfrac{1}{2}\mathrm{a}_u \bullet \mathrm{c}_u - \tfrac{1}{2}\mathrm{c}_u \iota_+ \tfrac{1+S}{1-S} \bullet \mathrm{c}_u$$

Thank you for your attention !

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