Around Van den Bergh's double brackets

Maxime Fairon

Laboratoire de Mathématiques d'Orsay
Université Paris Saclay

European Non-Associative Algebra Seminar
10/04/2023

Motivation: Kontsevich-Rosenberg principle

Fix (unital) associative algebra A over field $\mathbb{k}$ (char($\mathbb{k}$)$= 0$)

For $n \geq 1$, n-th representation space $\mathrm{Rep}_n(A)$ is scheme with B-points

$$\mathrm{Rep}_n(A)(B) := \mathrm{Hom}_{Alg.}(A, \mathrm{Mat}(n \times n, B))$$

$\mathbb{k}[\mathrm{Rep}_n(A)]$ is generated by 'matrix' symbols a_{ij} for $a \in A$, $1 \leq i, j \leq n$

Motivation: Kontsevich-Rosenberg principle

Fix (unital) associative algebra A over field $\Bbbk$ (char($\Bbbk$)$= 0$)

For $n \geq 1$, n-th representation space $\mathrm{Rep}_n(A)$ is scheme with B-points

$$\mathrm{Rep}_n(A)(B) := \mathrm{Hom}_{Alg.}(A, \mathrm{Mat}(n \times n, B))$$

$\Bbbk[\mathrm{Rep}_n(A)]$ is generated by 'matrix' symbols a_{ij} for $a \in A$, $1 \leq i, j \leq n$

Motto: [Kontsevich-Rosenberg,'00]

"A noncommutative structure of some kind on A should give an analogous commutative structure on all schemes $\mathrm{Rep}_n(A)$, $n \geq 1$."

structure $\mathcal{S}_{nc}$ in $Alg.$ ($e.g.$ formally smooth) $\longrightarrow$ structure $\mathcal{S}$ in $Com.Alg.$ ($e.g.$ smooth)

Van den Bergh's double brackets in 1 slide

$A^{\otimes 2} := A \otimes_{\Bbbk} A$, mult. $(a \otimes b)(c \otimes d) = ac \otimes bd$, swap $\tau_{(12)} a \otimes b = b \otimes a$.

Definition ([Van den Bergh, *double Poisson algebras*,'08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a, b\}\!\} = -\tau_{(12)} \{\!\{b, a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a, bc\}\!\} = (b \otimes 1) \{\!\{a, c\}\!\} + \{\!\{a, b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad, b\}\!\} = (1 \otimes a) \{\!\{d, b\}\!\} + \{\!\{a, b\}\!\} (d \otimes 1)$ (inner derivation)

Van den Bergh's double brackets in 1 slide

$A^{\otimes 2} := A \otimes_{\Bbbk} A$, mult. $(a \otimes b)(c \otimes d) = ac \otimes bd$, swap $\tau_{(12)} a \otimes b = b \otimes a$.

Definition ([Van den Bergh, *double Poisson algebras*,'08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1) \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a) \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} (d \otimes 1)$ (inner derivation)

Proposition ([Van den Bergh,'08])

If A is endowed with a double bracket $\{\!\{-,-\}\!\}$, then $\mathrm{Rep}_n(A)$ admits a unique $\mathrm{GL}_n(\Bbbk)$-invariant antisymmetric biderivation $\{-,-\}$ satisfying

$$\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj} \, \{\!\{a,b\}\!\}''_{il} \,. \tag{1}$$

(We write $\{\!\{a,b\}\!\} =: \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}'' \in A^{\otimes 2}$)

Moreover "double Jacobi identity" for $\{\!\{-,-\}\!\} \Rightarrow \{-,-\}$ is Poisson

Why should we care?

Double brackets are ...

- a starting point for noncommutative Poisson geometry
- related to other important algebraic structures
- useful in the study of integrable systems
- ...

Plan for the talk

1. **Double brackets and related structures**
2. Changing the derivation rules

Double Poisson brackets

Definition ([VdB,'08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1)\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}(1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a)\{\!\{d,b\}\!\} + \{\!\{a,b\}\!\}(d \otimes 1)$ (inner derivation)

We write: $\mathbb{J}\mathrm{ac} : A^{\otimes 3} \to A^{\otimes 3}$,

$\mathbb{J}\mathrm{ac} = \sum_{s \in \mathbb{Z}_3} \tau_{(123)}^{s} \circ (\{\!\{-,-\}\!\} \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \{\!\{-,-\}\!\}) \circ \tau_{(123)}^{-s}$

Double Poisson brackets

Definition ([VdB,'08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1) \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a) \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} (d \otimes 1)$ (inner derivation)

We write: $\mathbb{J}\mathrm{ac} : A^{\otimes 3} \to A^{\otimes 3}$,
$\mathbb{J}\mathrm{ac} = \sum_{s\in\mathbb{Z}_3} \tau_{(123)}^{s} \circ (\{\!\{-,-\}\!\} \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \{\!\{-,-\}\!\}) \circ \tau_{(123)}^{-s}$

Definition ([VdB,'08])

A double bracket $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ on A is a *double Poisson bracket* if $\mathbb{J}\mathrm{ac} \equiv 0$.

Explicitly, $\mathbb{J}\mathrm{ac}(a,b,c) = \{\!\{a, \{\!\{b,c\}\!\}'\}\!\} \otimes \{\!\{b,c\}\!\}''$
$+\tau_{(123)} \{\!\{b, \{\!\{c,a\}\!\}'\}\!\} \otimes \{\!\{c,a\}\!\}'' + \tau_{(132)} \{\!\{c, \{\!\{a,b\}\!\}'\}\!\} \otimes \{\!\{a,b\}\!\}''$

Double Poisson brackets: Examples (1)

Proposition ([VdB,'08])

If $(A, \{\!\{-,-\}\!\})$ *is a double Poisson algebra, then* $\mathrm{Rep}_n(A)$ *admits a unique* $\mathrm{GL}_n(\Bbbk)$*-inv. Poisson bracket s.t.* $\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}$, $\forall a, b \in A$

Example (Noncommutative $\mathfrak{gl}_n$)

$A = \Bbbk[x]$ with $\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$

$\mathrm{Rep}_n(A) \simeq \{X \in \mathrm{Mat}_{n\times n}(\Bbbk)\}$ with elementary functions $x_{ij}(X) := X_{ij}$

$\Rightarrow \{x_{ij}, x_{kl}\} = x_{kj}\delta_{il} - \delta_{kj}x_{il}$ (Lie-Poisson bracket of $\mathfrak{gl}_n$)

Example (Noncommutative $T^*\mathfrak{gl}_n$)

$A = \Bbbk\langle x, y\rangle$ with $\{\!\{x,y\}\!\} = 1 \otimes 1$, $\{\!\{x,x\}\!\} = 0$, $\{\!\{y,y\}\!\} = 0$

$\mathrm{Rep}_n(A) \simeq \{(X,Y) \in \mathrm{Mat}_{n\times n}(\Bbbk) \times \mathrm{Mat}_{n\times n}(\Bbbk)\}$

$\Rightarrow \{x_{ij}, y_{kl}\} = \delta_{kj}\delta_{il}$ and $\{x_{ij}, x_{kl}\} = 0 = \{y_{ij}, y_{kl}\}$ (symplectic PB)

Double Poisson brackets: Examples (2)

The previous examples are of the form $A = \mathrm{Ass}(V)$ for vector space V

Observation: if V is a *double Lie algebra* (= dPA without deriv. rules)
then A is a double Poisson algebra

Double Poisson brackets: Examples (2)

The previous examples are of the form $A = \mathrm{Ass}(V)$ for vector space V

Observation: if V is a *double Lie algebra* (= dPA without deriv. rules)
then A is a double Poisson algebra

Proposition ([Schedler,'09] [Odesskii-Rubtsov-Sokolov,'13] [Goncharov-Kolesnikov,'18])

The following are equivalent:

- *A double Lie bracket on* $V \simeq \mathbb{k}^n$
- *A skew-symmetric solution to AYBE on* $\mathrm{Mat}(n \times n, \mathbb{k})$
- *A skew-symmetric Rota-Baxter operator on* $\mathrm{Mat}(n \times n, \mathbb{k})$

$\rightsquigarrow$ classification of solutions to $AYBE$ and RB op. give examples

Double Poisson brackets to Leibniz brackets

Associated bracket

$$[-,-] = \mathrm{m} \circ \{\!\{-,-\}\!\} : A \times A \to A\,, \qquad [a,b] = \{\!\{a,b\}\!\}' \, \{\!\{a,b\}\!\}''$$

Double Poisson brackets to Leibniz brackets

Associated bracket
$[-,-] = \mathrm{m} \circ \{\!\{-,-\}\!\} : A \times A \to A\,, \qquad [a,b] = \{\!\{a,b\}\!\}' \, \{\!\{a,b\}\!\}''$

Lemma ([VdB,'08])

$[a,[b,c]] - [[a,b],c] - [b,[a,c]] = \mathrm{m} \circ (\mathrm{m} \otimes \mathrm{Id}_A)(\mathbb{J}\mathrm{ac}(a,b,c) - \mathbb{J}\mathrm{ac}(b,a,c))$

$\rightsquigarrow$ $[-,-]$ is a *left* Leibniz bracket if $\{\!\{-,-\}\!\}$ is Poisson

Double Poisson brackets to Leibniz brackets

Associated bracket

$[-,-] = \mathrm{m} \circ \{\!\{-,-\}\!\} : A \times A \to A\,, \qquad [a,b] = \{\!\{a,b\}\!\}' \, \{\!\{a,b\}\!\}''$

Lemma ([VdB,'08])

$[a,[b,c]] - [[a,b],c] - [b,[a,c]] = \mathrm{m} \circ (\mathrm{m} \otimes \mathrm{Id}_A)(\mathbb{J}\mathrm{ac}(a,b,c) - \mathbb{J}\mathrm{ac}(b,a,c))$

$\rightsquigarrow$ $[-,-]$ is a *left* Leibniz bracket if $\{\!\{-,-\}\!\}$ is Poisson

Additional properties:

1. $[a,-] \in \mathrm{Der}(A)$ for any $a \in A$
2. $[ab - ba, -] = 0$ for any $a,b \in A$

Double Poisson brackets to Lie brackets

Associated bracket $[-,-] = \mathrm{m} \circ \{\!\{-,-\}\!\} : A \times A \to A$ is Leibniz bracket

- $[ab - ba, c] = 0$ for any $a, b, c \in A$,
- $[a, -] \in \mathrm{Der}(A)$

Denote $H_0(A) := A/[A, A]$ (vector space!) $\quad A \ni a \mapsto a_\sharp \in H_0(A)$

Lemma ([VdB,'08])

$[-,-]$ *descends to antisym. map* $[-,-]_\sharp : H_0(A) \times H_0(A) \to H_0(A)$

Double Poisson brackets to Lie brackets

Associated bracket $[-,-] = \mathrm{m} \circ \{\!\{-,-\}\!\} : A \times A \to A$ is Leibniz bracket

- $[ab - ba, c] = 0$ for any $a, b, c \in A$,
- $[a, -] \in \mathrm{Der}(A)$

Denote $H_0(A) := A/[A, A]$ (vector space!) $\quad A \ni a \mapsto a_\sharp \in H_0(A)$

Lemma ([VdB,'08])

$[-,-]$ *descends to antisym. map* $[-,-]_\sharp : H_0(A) \times H_0(A) \to H_0(A)$

$\Rightarrow \quad (H_0(A), [-,-]_\sharp)$ is a Lie algebra.

Double Poisson brackets to Lie brackets

Associated bracket $[-,-] = \mathrm{m} \circ \{\!\{-,-\}\!\} : A \times A \to A$ is Leibniz bracket

- $[ab - ba, c] = 0$ for any $a, b, c \in A$,
- $[a, -] \in \mathrm{Der}(A)$

Denote $H_0(A) := A/[A, A]$ (vector space!) $\quad A \ni a \mapsto a_\sharp \in H_0(A)$

Lemma ([VdB,'08])

$[-,-]$ *descends to antisym. map* $[-,-]_\sharp : H_0(A) \times H_0(A) \to H_0(A)$

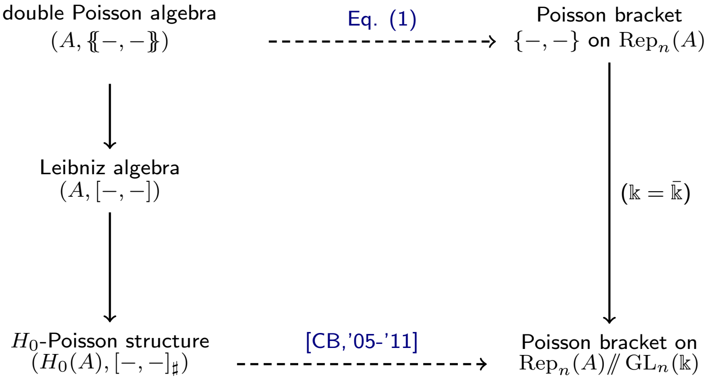
$\Rightarrow \quad (H_0(A), [-,-]_\sharp)$ is a Lie algebra.

Definition ([Crawley-Boevey,'05-'11])

A H_0*-Poisson structure* is a Lie bracket $[-,-]_\sharp$ on $H_0(A)$ such that each $[a_\sharp, -]_\sharp$ lifts to some $[a, -] \in \mathrm{Der}(A)$.

$$\Rightarrow \underbrace{(A, \{\!\{-,-\}\!\})}_{\text{double Poisson}} \rightsquigarrow \underbrace{(A, [-,-])}_{\text{Leibniz}} \rightsquigarrow \underbrace{(H_0(A), [-,-]_\sharp)}_{H_0\text{-Poisson}}$$

Double Poisson brackets and H_0-Poisson structures



Double Poisson brackets and H_0-Poisson structures

double Poisson algebra
$(A, \{\!\{-,-\}\!\})$
moment map $\mu \in A$

Eq. (1) ⇢

Poisson bracket
$\{-,-\}$ on $\mathrm{Rep}_n(A)$
mom. map $\hat{\mu}$ to $\mathfrak{gl}_n(\Bbbk)$

↓

Leibniz algebra
$(A, [-,-])$

↓

$(\Bbbk = \bar{\Bbbk})$ ↓

H_0-Poisson structure
$(H_0(A), [-,-]_\sharp)$
$H_0(A/\langle \mu - \lambda \mathrm{id}_A \rangle)$
fixed $\lambda \in \Bbbk$

[CB,'05-'11] ⇢

Poisson bracket on
$\mathrm{Rep}_n(A) /\!\!/ \mathrm{GL}_n(\Bbbk)$
$\hat{\mu}^{-1}(\lambda\, \mathrm{Id}_n) /\!\!/ \mathrm{GL}_n(\Bbbk)$

Examples: affine quiver varieties from $A = \Bbbk\overline{Q}$, $\mu = \sum_{a \in \overline{Q}} \epsilon(a)\, aa^*$ (over base ring!)

Poisson bracket on $\operatorname{Rep}_n(A)/\!\!/\operatorname{GL}_n(\mathbb{k})$

double Poisson $(A, \{\!\{-,-\}\!\})$ $\rightsquigarrow$ Poisson $(\operatorname{Rep}_n(A)/\!\!/\operatorname{GL}_n(\mathbb{k}), \{-,-\})$

Which variants induce a Poisson bracket on $\operatorname{Rep}_n(A)/\!\!/\operatorname{GL}_n(\mathbb{k})$?

Poisson bracket on $\mathrm{Rep}_n(A) /\!\!/ \mathrm{GL}_n(\Bbbk)$

double Poisson $(A, \{\!\!\{-,-\}\!\!\}) \rightsquigarrow$ Poisson $(\mathrm{Rep}_n(A) /\!\!/ \mathrm{GL}_n(\Bbbk), \{-,-\})$

Which variants induce a Poisson bracket on $\mathrm{Rep}_n(A) /\!\!/ \mathrm{GL}_n(\Bbbk)$?

- double *quasi*-Poisson bracket [VdB,'08]
  (relaxing $\mathbb{J}\mathrm{ac} = 0$)

Poisson bracket on $\operatorname{Rep}_n(A)/\!\!/\operatorname{GL}_n(\Bbbk)$

double Poisson $(A, \{\!\{-,-\}\!\}) \rightsquigarrow$ Poisson $(\operatorname{Rep}_n(A)/\!\!/\operatorname{GL}_n(\Bbbk), \{-,-\})$

Which variants induce a Poisson bracket on $\operatorname{Rep}_n(A)/\!\!/\operatorname{GL}_n(\Bbbk)$?

- double *quasi*-Poisson bracket [VdB,'08]
  (relaxing $\mathbb{J}\mathrm{ac} = 0$)
- *modified* double Poisson bracket [Arthamonov,'17]
  (skewsymmetry + Jacobi identity "up to commutators")
  $\hookrightarrow$ from λ-skewsym. Rota-Baxter operators [Goncharov-Gubarev,'22]

Poisson bracket on $\operatorname{Rep}_n(A)/\!\!/ \operatorname{GL}_n(\Bbbk)$

double Poisson $(A, \{\!\{-,-\}\!\}) \rightsquigarrow$ Poisson $(\operatorname{Rep}_n(A)/\!\!/ \operatorname{GL}_n(\Bbbk), \{-,-\})$

Which variants induce a Poisson bracket on $\operatorname{Rep}_n(A)/\!\!/ \operatorname{GL}_n(\Bbbk)$?

- double *quasi*-Poisson bracket [VdB,'08]
  (relaxing $\mathbb{J}\mathrm{ac} = 0$)
- *modified* double Poisson bracket [Arthamonov,'17]
  (skewsymmetry + Jacobi identity "up to commutators")
  $\hookrightarrow$ from λ-skewsym. Rota-Baxter operators [Goncharov-Gubarev,'22]
- Modifying the Leibniz rules [F.-McCulloch,'23]
- Also pre-Calabi-Yau algebras, ... but a whole different story!

Plan for the talk

1. Double brackets and related structures
2. **Changing the derivation rules**

Rewriting double brackets

Consider the following A-bimodule structures on $A \otimes A$

Outer bimodule: $a \cdot_{out} (d' \otimes d'') \cdot_{out} b \;=\; ad' \otimes d''b$

Inner bimodule: $a \cdot_{in} (d' \otimes d'') \cdot_{in} b \;=\; d'b \otimes ad''$

Definition ([Van den Bergh, *double Poisson algebras*,'08])

A double bracket on A is a $\mathbb{k}$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = b \cdot_{out} \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} \cdot_{out} c$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = a \cdot_{in} \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} \cdot_{in} d$ (inner derivation)

Rewriting double brackets

Consider the following A-bimodule structures on $A \otimes A$
Outer bimodule: $a \cdot_{out} (d' \otimes d'') \cdot_{out} b \;=\; ad' \otimes d''b$
Inner bimodule: $a \cdot_{in} (d' \otimes d'') \cdot_{in} b \;=\; d'b \otimes ad''$

Definition ([Van den Bergh, *double Poisson algebras*, '08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)}\, \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = b \cdot_{out} \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} \cdot_{out} c$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = a \cdot_{in} \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} \cdot_{in} d$ (inner derivation)

Observation I: 1+2 $\Rightarrow$ 3 because $\cdot_{in}$ **is the swap of** $\cdot_{out}$:

$$a \cdot_{in} (d' \otimes d'') \cdot_{in} b = \tau_{(12)} \big(a \cdot_{out} (\tau_{(12)}\, d' \otimes d'') \cdot_{out} b \big)$$

Rewriting double brackets

Consider the following A-bimodule structures on $A \otimes A$

Outer bimodule: $a \cdot_{out} (d' \otimes d'') \cdot_{out} b \ = \ ad' \otimes d''b$

Inner bimodule: $a \cdot_{in} (d' \otimes d'') \cdot_{in} b \ = \ d'b \otimes ad''$

Definition ([Van den Bergh, *double Poisson algebras*,'08])

A double bracket on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a, b\}\!\} = -\tau_{(12)} \ \{\!\{b, a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a, bc\}\!\} = b \cdot_{out} \{\!\{a, c\}\!\} + \{\!\{a, b\}\!\} \cdot_{out} c$ (outer derivation)
3. $\{\!\{ad, b\}\!\} = a \cdot_{in} \{\!\{d, b\}\!\} + \{\!\{a, b\}\!\} \cdot_{in} d$ (inner derivation)

Observation I: 1+2 $\Rightarrow$ 3 because $\cdot_{in}$ **is the swap of** $\cdot_{out}$:

$$a \cdot_{in} (d' \otimes d'') \cdot_{in} b = \tau_{(12)} \big(a \cdot_{out} (\tau_{(12)} \, d' \otimes d'') \cdot_{out} b\big)$$

Observation II: well-def. derivations because $\cdot_{out}$ is **swap-commuting**

$$a_i \cdot_{in} (a_o \cdot_{out} \ d' \otimes d'' \ \cdot_{out} b_o) \cdot_{in} b_i = a_o \cdot_{out} (a_i \cdot_{in} \ d' \otimes d'' \ \cdot_{in} b_i) \cdot_{out} b_o$$

Condition on bimodules

Definition (for bimodule denoted $\cdot$ (i.e. A-bimodule on $A \otimes A$))

The *swap* of $\cdot$ is the bimodule $*$ defined through

$$a * (d' \otimes d'') * b = \tau_{(12)}\big(a \cdot (\tau_{(12)}\, d' \otimes d'') \cdot b\big)$$

The bimodule $\cdot$ is *swap-commuting* if $\cdot$ commutes with $*$:

$$a_1 \cdot (a_2 * \ d' \otimes d'' \ * b_2) \cdot b_1 = a_2 * (a_1 \cdot \ d' \otimes d'' \ \cdot b_1) * b_2$$

Condition on bimodules

Definition (for bimodule denoted $\cdot$ (i.e. A-bimodule on $A \otimes A$))

The *swap* of $\cdot$ is the bimodule $*$ defined through

$$a * (d' \otimes d'') * b = \tau_{(12)}\big(a \cdot (\tau_{(12)}\, d' \otimes d'') \cdot b\big)$$

The bimodule $\cdot$ is *swap-commuting* if $\cdot$ commutes with $*$:

$$a_1 \cdot (a_2 * \ d' \otimes d'' \ * b_2) \cdot b_1 = a_2 * (a_1 \cdot \ d' \otimes d'' \ \cdot b_1) * b_2$$

Example

$$a \cdot_l d \cdot_l b = ad'b \otimes d''\,, \qquad \text{(left bimodule structure);}$$

$$a \cdot_r d \cdot_r b = d' \otimes ad''b\,, \qquad \text{(right bimodule structure);}$$

$$a \cdot_{out} d \cdot_{out} b = ad' \otimes d''b\,, \qquad \text{(outer bimodule structure);}$$

$$a \cdot_{in} d \cdot_{in} b = d'b \otimes ad''\,, \qquad \text{(inner bimodule structure).}$$

Condition on bimodules

Definition (for bimodule denoted $\cdot$ (i.e. A-bimodule on $A \otimes A$))

The *swap* of $\cdot$ is the bimodule $*$ defined through

$$a * (d' \otimes d'') * b = \tau_{(12)}\big(a \cdot (\tau_{(12)}\, d' \otimes d'') \cdot b\big)$$

The bimodule $\cdot$ is *swap-commuting* if $\cdot$ commutes with $*$:

$$a_1 \cdot (a_2 * \; d' \otimes d'' \; * b_2) \cdot b_1 = a_2 * (a_1 \cdot \; d' \otimes d'' \; \cdot b_1) * b_2$$

Example

$$a \cdot_l d \cdot_l b = ad'b \otimes d'' \,, \qquad \text{(left bimodule structure);}$$

$$a \cdot_r d \cdot_r b = d' \otimes ad''b \,, \qquad \text{(right bimodule structure);}$$

$$a \cdot_{out} d \cdot_{out} b = ad' \otimes d''b \,, \qquad \text{(outer bimodule structure);}$$

$$a \cdot_{in} d \cdot_{in} b = d'b \otimes ad'' \,, \qquad \text{(inner bimodule structure).}$$

- If $\cdot$ is swap-commuting, then $*$ is also
- Can "twist" by $\alpha, \beta \in \mathrm{Aut}(A)$, e.g. $a \cdot_l^{\alpha,\beta} d \cdot_l^{\alpha,\beta} b = \alpha(a) d' \beta(b) \otimes d''$

(Generalized) double brackets

From now on, we follow [F.-McCulloch,'23]

Definition

Fix a swap-commuting A-bimodule $\cdot$ with swap $*$

A double bracket (associated with $\cdot$) on A is a $\mathbb{k}$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)}\,\{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = b \cdot \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} \cdot c$ (derivation on the right by $\cdot$)
3. $\{\!\{ad,b\}\!\} = a * \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} * d$ (derivation on the left by $*$)

(Generalized) double brackets

From now on, we follow [F.-McCulloch,'23]

Definition

Fix a swap-commuting A-bimodule $\cdot$ with swap $*$
A double bracket (associated with $\cdot$) on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ with

1. $\{\!\{a,b\}\!\} = -\tau_{(12)}\ \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = b \cdot \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} \cdot c$ (derivation on the right by $\cdot$)
3. $\{\!\{ad,b\}\!\} = a * \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} * d$ (derivation on the left by $*$)

Take $\cdot = \cdot_{out} \rightsquigarrow$ Van den Bergh's theory from Part I

Remark: can go from $\{\!\{-,-\}\!\}$ to $\tau_{(12)} \circ \{\!\{-,-\}\!\}$ by replacing $\cdot$ with $*$
$\Rightarrow$ "equivalent" theories for double brackets associated with $\cdot$ and $*$

The inner case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with inner bimodule $\cdot_{in}$
$\rightsquigarrow$ expect to get analogues of VdB's results

The inner case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with inner bimodule $\cdot_{in}$ $\rightsquigarrow$ expect to get analogues of VdB's results

---

- $\mathbb{J}\mathrm{ac}$ "behaves well" $\rightsquigarrow$ double *Poisson* bracket when $\mathbb{J}\mathrm{ac} \equiv 0$

The inner case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with inner bimodule $\cdot_{in}$
$\rightsquigarrow$ expect to get analogues of VdB's results

- $\mathbb{J}\mathrm{ac}$ "behaves well" $\rightsquigarrow$ double *Poisson* bracket when $\mathbb{J}\mathrm{ac} \equiv 0$

Proposition (For a double Poisson bracket associated with $\cdot_{in}$)

1. $[-,-]_{in} := \mathrm{m} \circ \{\!\{-,-\}\!\}$ *is a* **right** *Leibniz bracket such that* $[-,a]_{in} \in \mathrm{Der}(A)$ *and* $[-,ab-ba]_{in} = 0$
2. $[-,-]_{in}$ *descends to a Lie bracket* $[-,-]_{in,\sharp}$ *on* $H_0(A)$ $\rightsquigarrow$ H_0*-Poissons structure (in right entry) on* $H_0(A)$

The inner case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with inner bimodule $\cdot_{in}$
$\rightsquigarrow$ expect to get analogues of VdB's results

- $\mathbb{J}\mathrm{ac}$ "behaves well" $\rightsquigarrow$ double *Poisson* bracket when $\mathbb{J}\mathrm{ac} \equiv 0$

Proposition (For a double Poisson bracket associated with $\cdot_{in}$)

1. $[-,-]_{in} := \mathrm{m} \circ \{\!\{-,-\}\!\}$ *is a* **right** *Leibniz bracket such that* $[-,a]_{in} \in \mathrm{Der}(A)$ *and* $[-,ab-ba]_{in} = 0$
2. $[-,-]_{in}$ *descends to a Lie bracket* $[-,-]_{in,\sharp}$ *on* $H_0(A)$ $\rightsquigarrow$ H_0*-Poissons structure (in right entry) on* $H_0(A)$

Furthermore:

3. $\mathrm{Rep}_n(A)$ *admits a unique* $\mathrm{GL}_n(\Bbbk)$*-inv. Poisson bracket such that* $\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{il}\ \{\!\{a,b\}\!\}''_{kj},\ \forall a,b \in A.$
4. *It descends to a Poisson bracket on* $\mathrm{Rep}_n(A)/\!\!/\,\mathrm{GL}_n(\Bbbk)$

(Parts 1,3 have analogues in the twisted case $\cdot_{in}^{\alpha,\alpha}$, but technical)

Double Jacobi identity

Crucial fact needed in the outer/inner cases:

- $\mathbb{J}\mathrm{ac}$ is a derivation in each argument for some bimodule struct. on $A^{\otimes 3}$

When true: $\mathbb{J}\mathrm{ac} \equiv 0$ iff $\mathbb{J}\mathrm{ac}(a, b, c) = 0$ for generators $a, b, c \in A$

Double Jacobi identity

Crucial fact needed in the outer/inner cases:

- $\mathbb{J}\mathrm{ac}$ is a derivation in each argument for some bimodule struct. on $A^{\otimes 3}$

When true: $\mathbb{J}\mathrm{ac} \equiv 0$ iff $\mathbb{J}\mathrm{ac}(a,b,c) = 0$ for generators $a,b,c \in A$

Example ($\{\!\{-,-\}\!\}$ associated with *right* bimodule)

$A = \Bbbk\langle x,y\rangle$ and $\lambda \in \Bbbk^\times$. Double bracket associated with $\cdot_r$ for

$$\{\!\{x,x\}\!\} = 0, \quad \{\!\{y,y\}\!\} = 0, \quad \{\!\{x,y\}\!\} = \lambda\, 1 \otimes 1\,.$$

$\mathbb{J}\mathrm{ac}(x,x,x) = \mathbb{J}\mathrm{ac}(x,x,y) = \mathbb{J}\mathrm{ac}(x,y,y) = 0$ (also for $x \leftrightarrow y$)

BUT $\mathbb{J}\mathrm{ac}(x,x,y^2) = -2\lambda^2\, 1 \otimes 1 \otimes 1.$

$\rightsquigarrow$ may need a weaker form of double Jacobi identity

weak Jacobiator

Definition ($\{\!\{-,-\}\!\}$ for some swap-commuting bimodule)

Let $\sigma, \sigma' \in \{(12),(13),(23)\}$. The $[\sigma,\sigma']$-*weak double Jacobiator* of $\{\!\{-,-\}\!\}$ is the map $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} : A^{\times 3} \to A^{\otimes 3}$ given by

$$[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} \;=\; \mathbb{J}\mathrm{ac} \;-\; \tau_\sigma^{-1} \circ \mathbb{J}\mathrm{ac} \circ \tau_{\sigma'} \,.$$

(Here, τ_σ is natural permut. on $A^{\otimes 3}$. E.g. $\tau_{(13)} a_1 \otimes a_2 \otimes a_3 = a_3 \otimes a_2 \otimes a_1$)

Definition

If $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} \equiv 0$, say $\{\!\{-,-\}\!\}$ is a *double* $[\sigma,\sigma']$-*weak Poisson bracket*. (Called double σ-weak Poisson bracket if $\sigma = \sigma'$.)

weak Jacobiator

Definition ($\{\!\{-,-\}\!\}$ for some swap-commuting bimodule)

Let $\sigma, \sigma' \in \{(12), (13), (23)\}$. The $[\sigma, \sigma']$-*weak double Jacobiator* of $\{\!\{-,-\}\!\}$ is the map $[\sigma, \sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} : A^{\times 3} \to A^{\otimes 3}$ given by

$$[\sigma, \sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} \,=\, \mathbb{J}\mathrm{ac} \,-\, \tau_\sigma^{-1} \circ \mathbb{J}\mathrm{ac} \circ \tau_{\sigma'} \,.$$

(Here, τ_σ is natural permut. on $A^{\otimes 3}$. E.g. $\tau_{(13)} a_1 \otimes a_2 \otimes a_3 = a_3 \otimes a_2 \otimes a_1$)

Definition

If $[\sigma, \sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} \equiv 0$, say $\{\!\{-,-\}\!\}$ is a *double* $[\sigma, \sigma']$-*weak Poisson bracket*. (Called double σ-weak Poisson bracket if $\sigma = \sigma'$.)

Proposition (By cyclic symmetry: $\tau_{(123)} \circ [\sigma, \sigma']\mathrm{wk}\mathbb{J}\mathrm{ac} \circ \tau_{(123)}^{-1} = [\sigma, \sigma']\mathrm{wk}\mathbb{J}\mathrm{ac}$)

There are 3 classes of distinct double σ-weak Poisson bracket

1.a. *double (12)-weak Poisson brackets* *(when $\sigma = \sigma'$)*

1.b. *double $[(12), (13)]$-weak Poisson brackets*

2. *double $[(12), (23)]$-weak Poisson brackets*

The right case (1)

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with right bimodule $\cdot_r$
$\rightsquigarrow$ expect to get differences with VdB's outer case

The right case (1)

We consider a double bracket $\{\!\!\{-,-\}\!\!\}$ associated with right bimodule $\cdot_r$
$\rightsquigarrow$ expect to get <u>differences</u> with VdB's outer case

---

- $\mathbb{J}\mathrm{ac}$ "behaves badly" (not a derivation $A \to A^{\otimes 3}$ in each argument)

- $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac}$ "behaves well" for $\sigma = \sigma' = (12)$

$\rightsquigarrow$ double (12)-*weak Poisson* bracket when $(12)\mathrm{wk}\mathbb{J}\mathrm{ac} \equiv 0$

The right case (1)

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with right bimodule $\cdot_r$
$\rightsquigarrow$ expect to get differences with VdB's outer case

---

- $\mathbb{J}\mathrm{ac}$ "behaves badly" (not a derivation $A \to A^{\otimes 3}$ in each argument)

- $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac}$ "behaves well" for $\sigma=\sigma'=(12)$

$\rightsquigarrow$ double (12)-*weak Poisson* bracket when $(12)\mathrm{wk}\mathbb{J}\mathrm{ac}\equiv 0$

Proposition (For a double (12)-weak Poisson bracket associated with $\cdot_r$)

1. *The double bracket descends to maps*
$_\bullet\{\!\{-,-\}\!\} : H_0(A)\times A \to H_0(A)\otimes A, \quad \{\!\{-,-\}\!\}_\bullet : A\times H_0(A)\to A\otimes H_0(A),$
which are derivations in their "A" factor.

2. *Both maps descend to the same operation*
$$_\bullet\{\!\{-,-\}\!\}_\bullet : H_0(A)\times H_0(A)\to H_0(A)\otimes H_0(A)$$
which uniquely extends to a Poisson bracket on $Sym(H_0(A))$ if $\{\!\{-,-\}\!\}$ is a double (12)-weak Poisson bracket.

The right case (2)

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with right bimodule $\cdot_r$
Assume it is a double (12)-*weak Poisson* bracket.

Proposition (continued)

3. $\mathrm{Rep}_n(A)$ *admits a unique* $\mathrm{GL}_n(\Bbbk)$*-inv. Poisson bracket such that* $\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{ij}\ \{\!\{a,b\}\!\}''_{kl}$, $\forall a, b \in A$.

4. *It descends to a Poisson bracket on* $\mathrm{Rep}_n(A)/\!\!/\,\mathrm{GL}_n(\Bbbk)$

Remark. In matrix notations, the induced Poisson bracket on $\mathrm{Rep}_n(A)$ follows the conventions of the "tensor notation" $\{- \overset{\otimes}{,} -\}$ from mathematical physics (e.g. used in connection to r-matrices, ...)

The left case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with left bimodule $\cdot_l$
$\rightsquigarrow$ expect to get analogues of right case (because equivalent under swap)

[1]**Question.** Is there a bimodule structure on $A \otimes A$ for which the notion of double $[(12),(23)]$-weak Poisson brackets is meaningful?

The left case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with left bimodule $\cdot_l$
$\rightsquigarrow$ expect to get analogues of right case (because equivalent under swap)

- $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac}$ "behaves well" for $\sigma=(12)$, $\sigma'=(13)$

$\rightsquigarrow$ double $[(12),(13)]$-*weak Poisson* brackets are interesting[1] for $\cdot_l$

[1] **Question.** Is there a bimodule structure on $A\otimes A$ for which the notion of double $[(12),(23)]$-weak Poisson brackets is meaningful?

The left case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with left bimodule $\cdot_l$
$\rightsquigarrow$ expect to get analogues of right case (because equivalent under swap)

- $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac}$ "behaves well" for $\sigma=(12)$, $\sigma'=(13)$

$\rightsquigarrow$ double $[(12),(13)]$-*weak Poisson* brackets are interesting[1] for $\cdot_l$

Proposition (For a double $[(12),(13)]$-weak Poisson bracket assoc. with $\cdot_l$)

2'. *The double bracket descends to a map*

$${}_\bullet\{\!\{-,-\}\!\}_\bullet : H_0(A)\times H_0(A)\to H_0(A)\otimes H_0(A)$$

which uniquely extends to a Poisson bracket on $Sym(H_0(A))$.

[1] **Question.** Is there a bimodule structure on $A\otimes A$ for which the notion of double $[(12),(23)]$-weak Poisson brackets is meaningful?

The left case

We consider a double bracket $\{\!\{-,-\}\!\}$ associated with left bimodule $\cdot_l$
$\rightsquigarrow$ expect to get analogues of right case (because equivalent under swap)

- $[\sigma,\sigma']\mathrm{wk}\mathbb{J}\mathrm{ac}$ "behaves well" for $\sigma=(12)$, $\sigma'=(13)$

$\rightsquigarrow$ double $[(12),(13)]$-*weak Poisson* brackets are interesting[1] for $\cdot_l$

Proposition (For a double $[(12),(13)]$-weak Poisson bracket assoc. with $\cdot_l$)

2'. *The double bracket descends to a map*

$$_\bullet\{\!\{-,-\}\!\}_\bullet : H_0(A)\times H_0(A)\to H_0(A)\otimes H_0(A)$$

which uniquely extends to a Poisson bracket on $Sym(H_0(A))$.

3'. $\mathrm{Rep}_n(A)$ *admits a unique* $\mathrm{GL}_n(\Bbbk)$*-inv. Poisson bracket such that* $\{a_{ij},b_{kl}\}=\{\!\{a,b\}\!\}'_{kl}\ \{\!\{a,b\}\!\}''_{ij}$, $\forall a,b\in A$.

4'. *It descends to* $\mathrm{Rep}_n(A)/\!\!/\,\mathrm{GL}_n(\Bbbk)$.

[1] **Question.** Is there a bimodule structure on $A\otimes A$ for which the notion of double $[(12),(23)]$-weak Poisson brackets is meaningful?

Thank you for your attention

Maxime Fairon

maxime.fairon@universite-paris-saclay.fr

www.imo.universite-paris-saclay.fr/en/perso/maxime-fairon/double-brackets/