Geometric reduction of spin trigonometric RS systems

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Plan for the talk

1. **Motivation : CM system**
2. RS system – what is known
3. RS system – new real form

Motivation (1) : CM system

[Calogero,'71 – Sutherland,'71 – Moser,'75]

Calogero-Moser-Sutherland (CM) system with trigonometric potential:

$$\ddot{q}_i = \gamma^2 \sum_{j \neq i} \frac{\cot\left(\frac{1}{2}(q_i - q_j)\right)}{\sin^2\left(\frac{1}{2}(q_i - q_j)\right)}, \quad \text{for } i = 1, \ldots, n$$

(parameter $\gamma \neq 0$, real or complex. 'Positions' $e^{\mathrm{i}q_j}$ on S^1 or $\mathbb{C}^\times$)

Definition as an integrable Hamiltonian system:

$$H_{CM} = \frac{1}{2} \sum_{1 \leq i \leq n} p_i^2 + \sum_{1 \leq i < j \leq n} \frac{\gamma^2}{\sin^2\left(\frac{1}{2}(q_i - q_j)\right)}$$

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Question: What is the phase space?

Motivation (2) : CM system by reduction

Approach* from [Kazhdan,Kostant,Sternberg;'78]

master space

$$\mathcal{M}_1 = T^*\,\mathrm{U}(n) \simeq \{(g,J) \in \mathrm{U}(n) \times \mathfrak{u}(n)\}$$

$\gamma > 0$

Hamiltonian reduction

$$\mathcal{N}_1 = \{(g,J) \mid \exists v \in \mathbb{C}^n,\ J - g^{-1}Jg + \mathfrak{i} vv^\dagger = \mathfrak{i}\,2\gamma\,\mathrm{Id}_n\}$$

$\mathfrak{i} = \sqrt{-1}$

reduced space

$$\mathcal{C}_{n,1} = \mathcal{N}_1 / \,\mathrm{U}(n)$$

Integrable system from $\mathrm{tr}(J), \ldots, \mathrm{tr}(J^n)$

* Real case. Holomorphic case: replace $\mathfrak{u}(n), \mathrm{U}(n)$ by $\mathfrak{gl}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})$ [Wilson,'98]

Motivation (3) : spin CM system

CM system with *"vector" spins* based on [Gibbons-Hermsen,'84]

coordinates*: $(q_i, p_i, a_i^\alpha, b_i^\alpha)_{1\le i\le n}^{1\le\alpha\le d}$ with $2n$ constraints $\rightsquigarrow$ dim $= 2nd$

$$\ddot{q}_i = \sum_{j\neq i} f_{ij} f_{ji} \frac{\cot\left(\frac{1}{2}(q_i - q_j)\right)}{4\,\sin^2\left(\frac{1}{2}(q_i - q_j)\right)} \qquad (\text{for } i = 1,\dots,n \text{ and } 1 \le \alpha \le d)$$

$$\dot{a}_i^\alpha = \sum_{j\neq i} \frac{f_{ij}}{4\sin^2\left(\frac{1}{2}(q_i - q_j)\right)} a_j^\alpha\,, \quad \dot{b}_i^\alpha = -\sum_{j\neq i} \frac{f_{ji}}{4\sin^2\left(\frac{1}{2}(q_i - q_j)\right)} b_j^\alpha$$

(We set $f_{ij} = \sum_{1\le\alpha\le d} a_i^\alpha b_j^\alpha$. Constraints include $f_{jj} = 2\gamma$, $\gamma \neq 0$.)

$$H_{CM}^{spin} = \frac{1}{2} \sum_{1\le i\le n} p_i^2 + \sum_{1\le i<j\le n} \frac{f_{ij} f_{ji}}{4\sin^2\left(\frac{1}{2}(q_i - q_j)\right)}$$

*For the real case: $b_i^\alpha = \overline{a}_i^\alpha \in \mathbb{C}$ + reality of Poisson bracket.

Motivation (4) : spin CM system by reduction

Same idea as "without spins" (which corresponds to $d = 1$)

master space

$$\mathcal{M}_d = T^* \mathrm{U}(n) \times (\mathbb{C}^n)^{\times d}$$
$$\simeq \{(g, J, v_\alpha)_\alpha \mid g \in \mathrm{U}(n),\ J \in \mathfrak{u}(n),\ v_\alpha \in \mathbb{C}^n\}$$

$\gamma > 0$

Ham. red.

$$\mathcal{N}_d = \{(J, g, v_\alpha) \mid J - g^{-1} J g + \mathrm{i} \sum_{1 \le \alpha \le d} v_\alpha v_\alpha^\dagger = \mathrm{i}\, 2\gamma\, \mathrm{Id}_n\}$$

reduced space

$$\mathcal{C}_{n,d} = \mathcal{N}_d / \mathrm{U}(n)$$

Degenerate integrability: algebra of first integrals of $(\mathrm{tr}(J^k))_k$ of codim. n

Complex case: $\mathfrak{u}(n), \mathrm{U}(n) \rightsquigarrow \mathfrak{gl}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})$, and $(v, v^\dagger) \rightsquigarrow (v, w) \in T^*\mathbb{C}^n$.

<u>Note:</u> For "coadjoint orbit"-valued spins, see e.g. L. Fehér, L.-C. Li, N. Reshetikhin, ...

Motivation (5) : RS system

[Ruijsenaars-Schneider,'86] : "relativistic" version of CM system.

Ruijsenaars-Schneider (RS) system with trigonometric potential

$$H_{RS} = \sum_{1 \leq i \leq n} e^{p_i} \prod_{k \neq i} \frac{\sin(q_i - q_k + \gamma)}{\sin(q_i - q_k)}$$

("MacDonald form" of the complex Hamiltonian. $\gamma \neq 0$.)

Question: What are the analogues of the previous constructions?

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Trigonometric RS system : 2 important approaches

Trig. CM (no spins): Hamiltonian reduction of T^*G by G, $G = \mathrm{U}(n), \mathrm{GL}_n(\mathbb{C})$

Approaches for the trig. RS system:

	complex case	real case
Hamiltonian reduction (Poisson-Lie)	[Fock-Rosly,'99]	[Fehér-Klimcik,'11] (non-compact)
quasi-Hamiltonian reduction	[Oblomkov,'04] [Chalykh-F.,'17]	[Fehér-Klimcik,'12] (compact)

Note: reduction starts with a *finite-dimensional* master space

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We will outline the 'quasi-' case

Note: reduction starts with a *finite-dimensional* master space

General construction

$(G = \mathrm{U}(n)$ or $\mathrm{GL}_n(\mathbb{C})$ if real/complex.)

quasi-Poisson point of view [Alekseev–Kosmann-Schwarzbach–Meinrenken,'02]
(+ [Alekseev–Malkin–Meinrenken,'98]; [Boalch,'07]; ...)

$\mathcal{M}^{qP}$: (internally fused) quasi-Poisson double
$\mathcal{M}^{qP} \simeq G \times G$

- quasi-Poisson bracket
- $G \curvearrowright$ simult. conjugation
- action by quasi-Poisson morphisms
- Group-valued moment map $\Phi : \mathcal{M}^{qP} \to G$

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- Group-valued moment map $\Phi : \mathcal{M}^{qP} \to G$

$\mathcal{N} = \Phi^{-1}(\mathrm{C})$ minimal smooth subspace of $\mathcal{M}^{qP}$
$(\rightsquigarrow \mathrm{C} = q\,\mathrm{Id}_n +$rank1 for $q = e^{\mathrm{i}\gamma}$ or $q \in \mathbb{C}^\times$ generic)

$\mathcal{C}_n = \mathcal{N}/G$ smooth symplectic of dim. $2n$

$\rightsquigarrow \exists$ local coordinates $(q_i, p_i)_{i=1}^n$ giving interpretation as trig. RS system

Dynamics

If $(A, B) \in G \times G \simeq \mathcal{M}^{qP}$, consider

$$\mathfrak{H} := \{H \mid H(A, B) = h(A),\ h \in \mathcal{C}^\infty(\mathrm{U}(n))^{\mathrm{U}(n)}\}$$

$\rightsquigarrow$ $\mathfrak{H}$ defines an integrable system on $\mathcal{C}_n = \mathcal{N}/G$

Byproduct of the reduction: get *explicit* commuting flows on $\mathcal{M}^{qP}$.

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Example (Real case) $(A,B) \in \mathrm{U}(n) \times \mathrm{U}(n)$

$H_1 = \Re[\mathrm{tr}(A)]$, $\dot{A} = 0$, $\dot{B} = \frac{1}{2}B(A - A^{-1})$
$H_2 = \Im[\mathrm{tr}(A)]$, $\dot{A} = 0$, $\dot{B} = -\frac{\mathrm{i}}{2}B(A + A^{-1})$

$\forall H \in \mathfrak{H}$, $\dot{A} = 0$, $\dot{B} = -B\nabla h(A)$
$\Rightarrow A(t) = A_0$, $B(t) = B_0 \exp(-t\nabla h(A_0))$ $\rightsquigarrow$ *complete* flows

Spin trigo. RS system

Trigo. RS system with *spins* in the complex setting [Krichever-Zabrodin,'95]

coordinates: $(q_i, a_i^\alpha, c_i^\alpha)_{1\leq i\leq n}^{1\leq\alpha\leq d}$ with n constraints ⇝ dim$= 2nd$

$$\ddot{q}_i = \sum_{j\neq i} f_{ij} f_{ji} \left(V(q_i - q_j) - V(q_j - q_i)\right) \,, \qquad \text{for } i = 1, \ldots, n$$

$$\dot{a}_i^\alpha = \sum_{j\neq i} f_{ij} V(q_i - q_j)\, a_j^\alpha \,, \quad \dot{c}_i^\alpha = -\sum_{j\neq i} f_{ji} V(q_j - q_i)\, c_j^\alpha \,, \qquad 1 \leq \alpha \leq d$$

(We set $f_{ij} = \sum_{1\leq\alpha\leq d} a_i^\alpha c_j^\alpha$, $V(q) = \coth(q) - \coth(q+\gamma)$, $\gamma \neq 0$.)

Question 1: What is the Hamiltonian formulation?

(Partial conjecture in [Arutyunov-Frolov,'98].)

Question 2: What is (are) the real form(s)?

spin RS systems : 2 important approaches

	complex case	real case
Hamiltonian reduction (Poisson-Lie)	[Arutyunov-Olivucci,'20] (2)	[F.-Fehér-Marshall,'21] (3) (non-compact)
quasi-Hamiltonian reduction	[Chalykh-F.,'20] (1)	[F.-Fehér,'23] (4) (non-compact)

Today: continue focus on the quasi-Hamiltonian point of view

spin RS systems : quasi-Hamiltonian red. $/\mathbb{C}$ (1)

master space

$$\mathcal{M}_d^{qP} \subsetneq \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times (T^*\mathbb{C}^n)^{\times d}$$

$$\simeq \{(X, Y, V_\alpha, W_\alpha)_\alpha \mid X, Y \in \mathrm{GL}_n,\ V_\alpha^T, W_\alpha \in \mathbb{C}^n\}$$

$\gamma \in \mathbb{C}^\times$, $e^{k\gamma} \neq 1$

quasi-Ham. reduction

$$\mathcal{N}_d^{qP} = \{XYX^{-1}Y^{-1} \overrightarrow{\prod_{1\leq\alpha\leq d}} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = e^\gamma\, \mathrm{Id}_n\}$$

reduced space

$$\mathcal{C}_{n,d} = \mathcal{N}_d^{qP} / \mathrm{GL}_n(\mathbb{C})$$

Quasi-Poisson structure from $\{(V, W) \in T^*\mathbb{C}^n \mid \det(\mathrm{Id}_n + WV) \neq 0\}$ $\rightsquigarrow$ [Van den Bergh,'08].

Theorem ([Chalykh-F.,'20])

On open subset of $\mathcal{C}_{n,d}$*, there exist local coordinates so that*

- *equations of motion associated with Hamiltonian* $\mathrm{tr}(Y)$ *reproduce the equations of the spin trig. RS system of [Krichever-Zabrodin,'95];*
- *the Poisson bracket written in local coordinates allows to prove the conjecture of [Arutyunov-Frolov,'98].*

Flows before reduction:

$$\dot{X} = XY, \quad \dot{Y} = 0, \quad \dot{V}_\alpha = 0, \quad \dot{W}_\alpha = 0$$

Moreover: integrability (Liouville + degenerate)

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The missing piece $/\mathbb{R}$ (1)

From now on: following [F.-Fehér, arXiv:2302.14392]

Objective: quasi-Hamiltonian reduction from master space

$$\begin{aligned} \mathcal{M}_d^{qP} &= \mathrm{U}(n) \times \mathrm{U}(n) \times [???]^{\times d} \\ &\subsetneq \mathrm{U}(n) \times \mathrm{U}(n) \times (\mathbb{C}^n)^{\times d} \end{aligned}$$

complex case:

$$\begin{aligned} \mathcal{M}_d^{qP} &= \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times (\{(V,W) \mid 1 + VW \neq 0\})^{\times d} \\ &\subsetneq \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times (T^*\mathbb{C}^n)^{\times d} \end{aligned}$$

The missing piece $/\mathbb{R}$ (2)

For $x \in \mathbb{R} \setminus \{0\}$, block [???] is $\mathrm{D}(x) := \{v \in \mathbb{C}^n \mid |v|^2 < \frac{2\pi}{|x|}\} \subset \mathbb{C}^n \simeq T^*\mathbb{R}^n$

Theorem ([F.-Fehér,'23])

The following defines a quasi-Poisson bracket on $\mathrm{D}(x)$*:*

$$\{v_i, v_k\} = 0,\ \{\bar{v}_i, \bar{v}_k\} = 0, \quad \{v_i, \bar{v}_k\} = \frac{\mathtt{i}}{x}\delta_{ik} + \frac{\mathtt{i}}{2}\mathtt{b}(x|v|^2)\left[|v|^2\delta_{ik} - v_i\bar{v}_k\right] \,,$$

for left mult. of $\mathrm{U}(n)$ *and* $\mathtt{b}(t) = \cot\left(\frac{t}{2}\right) - \frac{2}{t}$.

Key remark: $\Phi : \mathrm{D}(x) \to \mathrm{U}(n)$, $\Phi(v) = \exp(\mathtt{i}xvv^\dagger)$ is moment map.

Two proofs:

$\rightsquigarrow$ directly from the definition

$\rightsquigarrow$ *exponentiation* of $(T^*\mathbb{R}^n, \{-,-\}_{\mathsf{can}})$ (in sense of [AKSM,'02]); corresponds to 2-form of [Hurtubise-Jeffrey-Sjamaar,'06]

spin RS systems : quasi-Hamiltonian red. $/\mathbb{R}$ (1)

master space

$$\mathcal{M}_d^{qP} = \mathrm{U}(n) \times \mathrm{U}(n) \times \mathrm{D}(x_1) \times \cdots \times \mathrm{D}(x_d)$$

$$\simeq \{(A, B, v_\alpha)_\alpha \mid A, B \in \mathrm{U}(n),\ v_\alpha \in \mathrm{D}(x_\alpha) \subset \mathbb{C}^n\}$$

$k\gamma \notin 2\pi\mathbb{Z}$

quasi-Ham. reduction

$$\mathcal{N}_d^{qP} = \{ABA^{-1}B^{-1} \overrightarrow{\prod_{1\leq\alpha\leq d}} \exp(\mathrm{i}x_\alpha v_\alpha v_\alpha^\dagger) = e^{\mathrm{i}\gamma}\,\mathrm{Id}_n\}$$

reduced space

$$\mathcal{C}_{n,d} = \mathcal{N}_d^{qP} / \mathrm{U}(n)$$

To construct: local coordinates for interpretation as real spin RS system

$\rightsquigarrow$ Complex approach of [Chalykh-F.,20] not working!

spin RS systems : quasi-Hamiltonian red. $/\mathbb{R}$ (2)

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Theorem ([F.-Fehér,'23])

- *$\mathcal{C}_{n,d}$ not compact (for $d > 1$)*
- *$\mathcal{C}_{n,d}$ is smooth when $e^{\mathrm{i}k\gamma} \neq 1$ for $k = 1, \ldots, n$*
- *On a n-dimensional cover of $\mathcal{C}_{n,d}$, the vector fields induced by the functions $\Re[\mathrm{tr}(A)], \Im[\mathrm{tr}(A)]$ can be combined (complex-linearly) to give equations of motion of the spin RS system of [Krichever-Zabrodin,'95]*

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What about integrability of $\mathfrak{H} := \{h(A) \mid h \in \mathcal{C}^\infty(\mathrm{U}(n))^{\mathrm{U}(n)}\}$???

Dynamics on the master phase space

$$\mathcal{M}_d := \mathcal{M}_d^{qP} = \mathrm{U}(n) \times \mathrm{U}(n) \times \mathrm{D}(x_1) \times \cdots \times \mathrm{D}(x_d) \ \ni (A, B, v_\alpha)$$

$\rightsquigarrow$ **Hamiltonian quasi-Poisson manifold**[1] constructed by fusion of

- internally fused double $\mathbb{D}(\mathrm{U}(n)) \simeq \mathrm{U}(n) \times \mathrm{U}(n)$
- d Hamiltonian quasi-Poisson balls $\mathrm{D}(x_\alpha)$

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[1] In fact, $\exists$ pencil of compatible such structures; all are quasi-Hamiltonian!

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Theorem ([F.-Fehér,'23])

Set $\mathfrak{H} := \{H \mid H(A, B, v_\alpha) = h(A),\ h \in \mathcal{C}^\infty(\mathrm{U}(n))^{\mathrm{U}(n)}\}$*.*
$\forall H \in \mathfrak{H}$*, the quasi-Hamiltonian vector field* X_H *preserves the* $\mathrm{U}(n)$*-equivariant smooth map* $\Psi : \mathcal{M}_d \to \mathcal{M}_d$ *defined by*

$$\Psi : (A, B, v_1, \ldots, v_d) \mapsto (A, BAB^{-1}, v_1, \ldots, v_d)$$

Moreover $\mathfrak{F} := \Psi^*(\mathcal{C}^\infty(\mathcal{M}_d))$ *is a ring of first integrals with functional dimension* $\dim(\mathcal{M}_d) - n \Rightarrow (\mathfrak{H}, \mathfrak{F})$ *is a "degenerate integrable system"*

[1] In fact, $\exists$ pencil of compatible such structures; all are quasi-Hamiltonian!

Dynamics on the reduced Poisson space (1)

The "interesting" slice $\mathcal{C}_{n,d} = \Phi^{-1}(e^{\mathrm{i}\gamma}\,\mathrm{Id}_n)/\,\mathrm{U}(n)$ lies inside $\mathcal{M}_d/\,\mathrm{U}(n)$

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Let $\mathcal{M}_{d*} \subset \mathcal{M}_d$ where $\mathrm{U}(n)$ acts freely. Set $\mathcal{M}_{d*}^{\mathrm{red}} := \mathcal{M}_{d*}/\,\mathrm{U}(n)$
($\mathcal{M}_{d*}^{\mathrm{red}}$ filled by symplectic leaves such as $\mathcal{C}_{n,d}$, but generic codimension is n)

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Let $\mathcal{M}_{d**} \subset \mathcal{M}_{d*}$ where (i) $\mathrm{U}(n)$ acts freely on (A, v_α); (ii) $A \in \mathrm{U}(n)_{\mathrm{reg}}$
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Set $\mathcal{M}_{d**}^{\mathrm{red}} := \mathcal{M}_{d**}/\,\mathrm{U}(n)$

Theorem ([F.-Fehér,'23])

$\forall H \in \mathfrak{H}$, the quasi-Hamiltonian vector field X_H restricted to $\mathcal{M}_{d**}^{\mathrm{red}}$ preserves the smooth submersion ψ_{red} induced on $\mathcal{M}_{d**}^{\mathrm{red}}$ after restricting $\Psi : \mathcal{M}_d \to \mathcal{M}_d$ to $\mathcal{M}_{d**}$.
Moreover $\mathfrak{F}_{\mathrm{red}} := \psi_{\mathrm{red}}^{*}\,\mathcal{C}^\infty(\Psi(\mathcal{M}_{d**})/\,\mathrm{U}(n))$ is a ring of first integrals with functional dimension $\dim(\mathcal{M}_{d**}^{\mathrm{red}}) - n$.

$\Rightarrow$ $(\mathfrak{H}_{\mathrm{red}}, \mathfrak{F}_{\mathrm{red}})$ *is a degenerate integrable system on* $\mathcal{M}_{d**}^{\mathrm{red}}$

Dynamics on the reduced Poisson space (2)

What about restricting this generic integrable system to symplectic leaves?

Dynamics on the reduced Poisson space (2)

What about restricting this generic integrable system to symplectic leaves?

1. This **works** for $(\Phi^{-1}(\mathrm{C}_{\mathrm{reg}}) \cap \mathcal{M}_{d**}) / \mathrm{U}(n) \subset \mathcal{M}_{d**}^{\mathrm{red}}$ where $\mathrm{C}_{\mathrm{reg}} \subset \mathrm{U}(n)_{\mathrm{reg}}$
2. Not yet possible on $(\Phi^{-1}(e^{\mathrm{i}\gamma}\,\mathrm{Id}_n) \cap \mathcal{M}_{d**}) / \mathrm{U}(n) = \mathcal{C}_{n,d} \cap \mathcal{M}_{d**}^{\mathrm{red}}$

Thank you for your attention !

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