Compatible Poisson structures on multiplicative quiver varieties

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Plan for the talk

1. **Motivation : RS system and a conjecture**
2. Poisson pencil for quiver varieties
3. Poisson pencil for multiplicative quiver varieties
4. Proving the conjecture

Motivation (1) : CM system

[Calogero,'71 – Sutherland,'71 – Moser,'75]

Calogero-Moser-Sutherland (CM) system with trigonometric potential:

$$\ddot{q}_i = \gamma^2 \sum_{j \neq i} \frac{\cot\left(\frac{1}{2}(q_i - q_j)\right)}{\sin^2\left(\frac{1}{2}(q_i - q_j)\right)}\,, \qquad \text{for } i = 1, \ldots, n$$

(Complex case: parameter $\gamma \in \mathbb{C}^\times$; 'Positions' $e^{\mathrm{i}q_j}$ on $\mathbb{C}^\times$)

Definition as an integrable Hamiltonian system:

$$H_{CM} = \frac{1}{2} \sum_{1 \leq i \leq n} p_i^2 + \sum_{1 \leq i < j \leq n} \frac{\gamma^2}{\sin^2\left(\frac{1}{2}(q_i - q_j)\right)}$$

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Phase space: Hamiltonian reduction of $T^* \mathrm{GL}_n(\mathbb{C})$ at $\mathcal{O}_{\mathrm{Id} + \text{rank } 1}$

Motivation (2) : CM-RS systems

The zoo of Calogero-Ruijsenaars integrable systems

real or complex system
($\dim_{\mathbb{R}}$ or $\dim_{\mathbb{C}}$)

'spin' degrees of freedom
($d \geq 2$, $\dim = 2nd$)

Calogero-Moser system
rational ($\dim = 2n$)

$$\ddot{q}_i = \sum_{j \neq i} \frac{-2\gamma^2}{(q_i - q_j)^3}$$

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...

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Ruijsenaars-Schneider system
(relativistic version)

other potentials
(trigo.-hyperbolic/elliptic)

relation to Lie algebras
($q_i - q_j = \mathbf{q} \cdot (\epsilon_i - \epsilon_j) \rightsquigarrow A_{n-1}$)

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Central question: what is the phase space?

Motivation (3) : spin RS systems

[Arutyunov-Frolov,'98]

- Construction of the spin, rational RS system by reduction of $T^* \mathrm{GL}_n(\mathbb{C}) \times (T^* \mathbb{C}^n)^d$
- Conjecture about some Poisson brackets in the spin, trigo. RS system

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[Chalykh-F.,'20; 1811.08727]

- Proof of the conjecture
  $\rightsquigarrow$ Phase space constructed as a *multiplicative quiver variety*
- Complete characterization of the Poisson bracket in local coordinates

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[Arutyunov-Olivucci,'20; 1906.02619]

- Phase space constructed using reduction by *Poisson-Lie symmetries*
- Complete characterization of the Poisson bracket in local coordinates
  $\rightsquigarrow$ different from [CF,'20]
  $\rightsquigarrow$ does not satisfy the conjecture of [AF,'98]

How to compare the 2 Poisson structures?

Motivation (4) : the Poisson brackets

[Chalykh-F.,'20] Coordinates $(x_i, a_i^\alpha, b_i^\alpha)_{1\leq i\leq n}^{1\leq\alpha\leq d}$ subject to $\sum_{\alpha=1}^d a_i^\alpha = 1$ (+ invert. cond.)

$$\{x_i, x_j\}_0 = 0\,, \quad \{x_i, a_j^\alpha\}_0 = 0\,, \quad \{x_i, b_j^\alpha\}_0 = \delta_{ij} x_i b_j^\alpha\,,$$

$$\begin{aligned}\{a_i^\alpha, a_j^\beta\}_0 =& \frac{1}{2}\delta_{(i\neq j)}\frac{x_i + x_j}{x_i - x_j}(a_i^\alpha a_j^\beta + a_j^\alpha a_i^\beta - a_j^\alpha a_j^\beta - a_i^\alpha a_i^\beta) + \frac{1}{2}o(\beta,\alpha)(a_i^\alpha a_j^\beta + a_j^\alpha a_i^\beta) \\ &+ \frac{1}{2}\sum_{\gamma=1}^d o(\alpha,\gamma) a_j^\beta (a_i^\alpha a_j^\gamma + a_j^\alpha a_i^\gamma) - \frac{1}{2}\sum_{\gamma=1}^d o(\beta,\gamma) a_i^\alpha (a_j^\beta a_i^\gamma + a_i^\beta a_j^\gamma)\, [+\ldots+\ldots],\end{aligned}$$

$$\begin{aligned}\{a_i^\alpha, b_j^\beta\}_0 =& a_i^\alpha Z_{ij} - \delta_{\alpha\beta} Z_{ij} - \frac{1}{2}\delta_{(i\neq j)}\frac{x_i + x_j}{x_i - x_j}(a_i^\alpha - a_j^\alpha) b_j^\beta + \delta_{(\alpha<\beta)} a_i^\alpha b_j^\beta \; [+\ldots+\ldots] \\ &+ a_i^\alpha \sum_{\gamma=1}^{\beta-1} a_i^\gamma (b_j^\gamma - b_j^\beta) - \delta_{\alpha\beta}\sum_{\gamma=1}^{\beta-1} a_i^\gamma b_j^\gamma - \frac{1}{2}\sum_{\gamma=1}^d o(\alpha,\gamma) b_j^\beta (a_i^\alpha a_j^\gamma + a_j^\alpha a_i^\gamma)\,,\end{aligned}$$

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for $Z_{ij} = q\dfrac{\sum_{\beta=1}^d a_i^\beta b_j^\beta}{x_i x_j^{-1} - q}$, $o(\alpha,\beta) = \text{sign}(\beta - \alpha)$

Red terms: missing in [Arutyunov-Olivucci,'20] – Blue terms: extra terms in [Arutyunov-Olivucci,'20]

Plan for the talk

Conjecture: The two Poisson structures are compatible
(cf. [F.-Fehér,'23; 2302.14392])

Approach: find them as compatible structures *before* quasi-Hamiltonian reduction of the multiplicative quiver variety

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1. Motivation : RS system and a conjecture
2. **Poisson pencil for (additive/Nakajima) quiver varieties**
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Quivers and representations

Quiver Q: directed graph (vertices $s \in I$, arrows $a \in Q$)

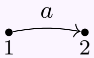
Path algebra $\mathbb{C}Q$:
generators are $a \in Q$ and $(e_s)_{s \in I}$
relations are $a = e_{t(a)} a e_{h(a)}$, $e_s e_t = \delta_{st} e_s$ [left-to-right convention!]

Representation space $\mathrm{Rep}(\mathbb{C}Q, \mathbf{n})$: (for $\mathbf{n} = (n_s)_{s \in I}$, n_s integer)

$a \mapsto \mathtt{X}(a) \in \mathrm{Mat}_{\mathbf{n} \times \mathbf{n}}(\mathbb{C})$, $e_s \mapsto \mathrm{Id}_{n_s} \subset \mathrm{Id}_{\mathbf{n}}$ (Rep. on $\mathbb{C}^{\mathbf{n}} = \oplus_s \mathbb{C}^{n_s}$)

Example

Q_-



$\mathbb{C}Q_-$

$\mathrm{Rep}(\mathbb{C}Q_-, (n_1, n_2))$

Quiver variety

double $\overline{Q}$: given Q, we add $h(a) \xrightarrow{a^*} t(a)$, for all $t(a) \xrightarrow{a} h(a) \in Q$

for $b \in \overline{Q}$, $\epsilon(b) = \begin{cases} +1 & b \in Q \\ -1 & b \in \overline{Q} \setminus Q \end{cases}$

Quiver variety $M(Q, \mathbf{n}, q)$: ($\mathbf{n} = (n_s)$ dim., $q = (q_s)$ parameter)

$$\operatorname{Rep}(\mathbb{C}\overline{Q}, \mathbf{n}) \supset \left\{ \sum_{b \in \overline{Q}} \epsilon(b)\, \mathtt{X}(b)\mathtt{X}(b^*) = q \cdot \operatorname{Id}_{\mathbb{C}^{\mathbf{n}}} \right\} =: \operatorname{Rep}(\Pi^q(Q), \mathbf{n})$$

$$M(Q, \mathbf{n}, q) := \operatorname{Rep}(\Pi^q(Q), \mathbf{n}) /\!/ \prod_s \operatorname{GL}(n_s)$$

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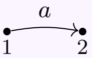
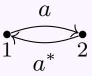


$\overline{Q}_-$



$\operatorname{Rep}(\mathbb{C}\overline{Q}_-, \mathbf{n})$

$\operatorname{Rep}(\Pi^q(Q_-), \mathbf{n})$

Poisson structure on QV

QV have a Poisson bracket using Hamiltonian reduction [Nakajima,'94]

Example

Recall $\mathrm{Rep}(\mathbb{C}\overline{Q}_-, \mathbf{n})$ is parametrised by
$A \in \mathrm{Mat}_{n_1 \times n_2}(\mathbb{C}), \quad B \in \mathrm{Mat}_{n_2 \times n_1}(\mathbb{C})$ (=nonzero blocks in $\mathtt{X}(a), \mathtt{X}(a^*)$)

The Poisson bracket is $\{A_{ij}, A_{kl}\} = 0 = \{B_{ij}, B_{kl}\}$, $\{A_{ij}, B_{kl}\} = \delta_{kj}\delta_{il}$

given by the bivector $P_{Q_-} := \mathrm{tr}[\partial_a \wedge \partial_{a^*}]$

The moment map is $(AB, -BA) \in \mathfrak{gl}(n_1) \times \mathfrak{gl}(n_2)$

The general case follows from Q_- by taking sums

Notation: ∂_a is matrix-valued vector field, where $(\partial_a)_{ij} = \partial / \partial \mathtt{X}(a)_{ji}$

Compatible Poisson structures on QV

Theorem (for Q general, [Nakajima,'94])

$\mathrm{Rep}(\mathbb{C}\overline{Q},\mathbf{n})$ *has the Poisson bivector* $P_{Q,\mathtt{qv}} := \sum_{a\in Q} \mathrm{tr}[\partial_a \wedge \partial_{a^*}]$ *and the moment map* $\mu((\mathtt{X}(a))_{a\in\overline{Q}}) = \sum_{a\in Q} [\mathtt{X}(a),\mathtt{X}(a^*)]$.
Moreover, the bivector is non-degenerate, and the corresponding symplectic form is $\omega_{Q,\mathtt{qv}} = -\sum_{a\in Q} \mathrm{tr}[d\mathtt{X}(a)\wedge d\mathtt{X}(a^*)]$

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Observation: $\forall b\in Q$, we have a $\mathbb{C}^\times$ action by Hamiltonian Poisson automorphisms: $\mathcal{A}_b : \lambda\cdot(\mathtt{X}(a))_{a\in\overline{Q}} = (\lambda^{\delta_{ab}-\delta_{a,b^*}}\mathtt{X}(a))_{a\in\overline{Q}}$

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Write $\mathrm{Inf}^{(b)}$ for infinitesimal action of $1\in\mathbb{C}=\mathrm{Lie}(\mathbb{C}^\times)$ through $\mathcal{A}_b$

Theorem (Folklore?)

Fix an ordering $<$ on Q. Take any $z_{ab}\in\mathbb{C}$ for any $a<b$. The above theorem is true for $P_{Q,\mathtt{qv}} + \sum_{a<b} z_{ab}\,\mathrm{Inf}^{(a)}\wedge\mathrm{Inf}^{(b)}$ with compatible 2-form $\omega_{Q,\mathtt{qv}} + \sum_{a<b} z_{ab}\,\mathrm{tr}[d(\mathtt{X}(a)\mathtt{X}(a^*))]\wedge\mathrm{tr}[d(\mathtt{X}(b)\mathtt{X}(b^*))]$

$\rightsquigarrow$ "pencil" descends to $M(Q,\mathbf{n},q)$ for any $\mathbf{n},q$

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Multiplicative quiver variety

Quiver variety $M(Q,\mathbf{n},q) := \mathrm{Rep}(\Pi^q(Q),\mathbf{n})/\!/ \prod_s \mathrm{GL}(n_s)$ $\quad$ ($\mathbf{n} = (n_s)$ dim., $q = (q_s) \in \mathbb{C}^{|I|}$)

$$\mathrm{Rep}(\Pi^q(Q),\mathbf{n}) := \left\{ \sum_{b\in\overline{Q}} \epsilon(b)\, \mathtt{X}(b)\mathtt{X}(b^*) = q \cdot \mathrm{Id}_{\mathbb{C}^{\mathbf{n}}} \right\} \subset \mathrm{Rep}(\mathbb{C}\overline{Q},\mathbf{n})$$

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Multiplicative quiver variety $\mathcal{M}(Q,\mathbf{n},q)$ $\quad$ ($\mathbf{n}=(n_s)$ dim., $q=(q_s)\in(\mathbb{C}^\times)^{|I|}$)

$$\mathrm{Rep}(\Lambda^q(Q),\mathbf{n}) := \left\{ \prod_{b\in\overline{Q}} (\mathrm{Id} + \mathtt{X}(b)\mathtt{X}(b^*))^{\epsilon(b)} = q\cdot \mathrm{Id}_{\mathbb{C}^{\mathbf{n}}} \right\} \subset \mathrm{Rep}(\mathbb{C}\overline{Q},\mathbf{n})$$

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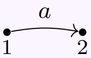
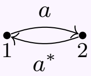


$\overline{Q}_-$



$\mathrm{Rep}(\mathbb{C}\overline{Q}_-,\mathbf{n})$

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Poisson structure on MQV (1)

MQV have a Poisson bracket [Van den Bergh,'08;Yamakawa,'08]

$\rightsquigarrow$ construction by quasi-Hamiltonian reduction [A-KS-M,'02]

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Comparing reductions for a space $\mathcal{N}$ with action of $G = \mathrm{GL}_n(\mathbb{C})$

Hamiltonian reduction	**quasi-Hamiltonian reduction**
• Poisson bracket on $\mathcal{N}$	• quasi-Poisson bracket on $\mathcal{N}$
• Jacobi identity	• Jacobi id. only for G-inv. functions
• action by Poisson morphism	• action by quasi-Poisson morphisms
• moment map $\Phi : \mathcal{N} \to \mathrm{Lie}(G)^*$	• moment map $\Phi : \mathcal{N} \to G$

$\mathcal{N}/\!\!/G$ and $\Phi^{-1}(\xi\,\mathrm{Id}_n)/\!\!/G$ have a Poisson bracket

Poisson structure on MQV (2)

Example (The case of $\overline{Q}_-$)

The quasi-Poisson bracket is $\{A_{ij}, A_{kl}\}_{\mathrm{P}} = 0$, $\{B_{ij}, B_{kl}\}_{\mathrm{P}} = 0$,
$\{A_{ij}, B_{kl}\}_{\mathrm{P}} = \delta_{kj}\delta_{il} + \frac{1}{2}(BA)_{kj}\delta_{il} + \frac{1}{2}\delta_{kj}(AB)_{il}$ given by bivector $P_{Q_-,\mathtt{mqv}}$

The moment map is (on dense open $\mathrm{Rep}^\circ(\mathbb{C}\overline{Q}_-, \mathbf{n}) \subset \mathrm{Rep}(\mathbb{C}\overline{Q}_-, \mathbf{n})$)
$\left(\mathrm{Id}_{n_1} + AB, (\mathrm{Id}_{n_2} + BA)^{-1}\right) \in \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$

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Theorem (Q general; ordering $<$, [Van den Bergh,'08;Yamakawa,'08])

$\mathrm{Rep}^\circ(\mathbb{C}\overline{Q}, \mathbf{n})$ *has the quasi-Poisson bivector* $P_{Q,\mathtt{mqv}}$ *given by*

$$\frac{1}{2}\sum_{a\in\overline{Q}} \epsilon(a)\ \mathrm{tr}\left[(\mathrm{Id} + \mathtt{X}(a^*)\mathtt{X}(a))\,\partial_a \wedge \partial_{a^*}\right] - \frac{1}{2}\sum_{a<b} \mathrm{tr}\left[(\partial_{a^*}\mathtt{X}(a^*) - \mathtt{X}(a)\,\partial_a) \wedge (\partial_{b^*}\mathtt{X}(b^*) - \mathtt{X}(b)\,\partial_b)\right]$$

and the moment map $\Phi((\mathtt{X}(a))_{a\in\overline{Q}}) = \prod_{a\in\overline{Q}} (\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))^{\epsilon(a)}$.
Moreover, the bivector is non-degenerate for the corresponding quasi-Hamiltonian form $\omega_{Q,\mathtt{mqv}}$ *given by*

$$-\frac{1}{2}\sum_{a\in\overline{Q}} \epsilon(a)\,\mathrm{tr}[(\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))^{-1} d\mathtt{X}(a) \wedge d\mathtt{X}(a^*)] - \frac{1}{2}\sum_{a\in\overline{Q}} \mathrm{tr}[\Phi_a^{-1}\, d\Phi_a \wedge d[\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*)]^{\epsilon(a)} (\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))^{-\epsilon(a)}]$$

Compatible Poisson structures on MQV

Recall: $\forall b \in Q$, $\mathbb{C}^\times$ action $\mathcal{A}_b : \lambda \cdot (\mathtt{X}(a))_{a\in\overline{Q}} = (\lambda^{\delta_{ab}-\delta_{a,b^*}}\mathtt{X}(a))_{a\in\overline{Q}}$

Write $\mathrm{Inf}^{(b)}$ for infinitesimal action of $1 \in \mathbb{C} = \mathsf{Lie}(\mathbb{C}^\times)$ through $\mathcal{A}_b$

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Theorem ([F.,arXiv:2310.18751])

Fix an ordering $<$ on Q. Take any $z_{ab} \in \mathbb{C}$ for any $a < b$.

The previous theorem is true for $P_{Q,\mathtt{mqv}} + \sum_{a<b} z_{ab}\, \mathrm{Inf}^{(a)} \wedge \mathrm{Inf}^{(b)}$

with compatible 2-form $\omega_{Q,\mathtt{mqv}} + \sum_{a<b} z_{ab}\, T(a) \wedge T(b)$

Here: $T(a) := \mathrm{tr}\left[(\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))^{-1}\, d(\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))\right]$

$\rightsquigarrow$ "pencil" descends to the MQV $\mathcal{M}(Q, \mathbf{n}, q)$ for any $\mathbf{n}, q$

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The previous theorem is true for $P_{Q,\mathtt{mqv}} + \sum_{a<b} z_{ab}\, \mathrm{Inf}^{(a)} \wedge \mathrm{Inf}^{(b)}$

with compatible 2-form $\omega_{Q,\mathtt{mqv}} + \sum_{a<b} z_{ab}\, T(a) \wedge T(b)$

Here: $T(a) := \mathrm{tr}\left[(\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))^{-1}\, d(\mathrm{Id} + \mathtt{X}(a)\mathtt{X}(a^*))\right]$

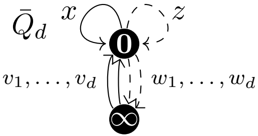
$\rightsquigarrow$ "pencil" descends to the MQV $\mathcal{M}(Q,\mathbf{n},q)$ for any $\mathbf{n}, q$

Note: Method works for character varieties or Boalch's generalized MQV

Plan for the talk

1. Motivation : RS system and a conjecture
2. Poisson pencil for quiver varieties
3. Poisson pencil for multiplicative quiver varieties
4. **Proving the conjecture**

Global construction of the phase space



$d \geq 2$ spins [Chalykh-F.,'20 / 1811.08727]

$\text{Rep}\left(\bar{Q}_d, (1, n)\right)$

parametrised by : $X, Z \in \mathfrak{gl}_n$

$V_\alpha \in \text{Mat}_{1\times n}, W_\alpha \in \text{Mat}_{n\times 1}$

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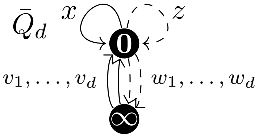

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$$\mathcal{C}^{\times}_{n,d,q} := \left\{ XZX^{-1}Z^{-1} \overrightarrow{\prod_{1\leq\alpha\leq d}} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\,\mathrm{Id}_n \right\} /\!/ \,\mathrm{GL}_n$$

Ruijsenaars-Schneider space with d spin/degrees of freedom

$\rightsquigarrow$ Poisson structure from Van den Bergh's bivector $P_{Q_d,\mathtt{mqv}}$

Local construction of the phase space

$$\mathcal{C}^{\times}_{n,d,q} := \Big\{ XZX^{-1}Z^{-1} \overrightarrow{\prod_{1\le\alpha\le d}} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\,\mathrm{Id}_n \Big\} /\!/ \,\mathrm{GL}_n$$

Let $A_\alpha := W_\alpha$, $B_\alpha := V_\alpha(\mathrm{Id}_n + W_{\alpha-1}V_{\alpha-1}) \cdots (\mathrm{Id}_n + W_1V_1)Z$, $(1 \le \alpha \le d)$

$\rightsquigarrow \mathcal{C}^{\times}_{n,d,q} := \Big\{ XZX^{-1} = qZ + q\sum_{1\le\alpha\le d} A_\alpha B_\alpha \Big\} /\!/ \,\mathrm{GL}_n$

Local construction of the phase space

$$\mathcal{C}^{\times}_{n,d,q} := \left\{ XZX^{-1}Z^{-1} \prod_{1\leq\alpha\leq d}^{\longrightarrow} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\,\mathrm{Id}_n \right\} /\!/ \,\mathrm{GL}_n$$

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Theorem ([Chalykh-F.,'20])

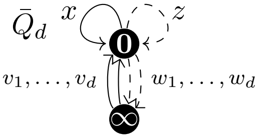
There exists coordinates $(x_i, a_i^\alpha, b_i^\alpha)^{1\leq\alpha\leq d}_{1\leq i\leq n}$ satisfying $\sum_{1\leq\alpha\leq d} a_i^\alpha = 1$ such that $\mathcal{C}^{\times}_{n,d,q}$ is parametrised by*

$$X_{ij} = \delta_{ij}x_i, \quad Z_{ij} = q\frac{\sum_{\beta=1}^d a_i^\beta b_j^\beta}{x_i x_j^{-1} - q}, \quad (A_\alpha)_i = a_i^\alpha, \quad (B_\alpha)_i = b_i^\alpha.$$

The Poisson bracket induced by $P_{Q_d,\mathtt{mqv}}$ is given by ▸ Intro Slide (p.10)*, which satisfies the conjecture of [Arutyunov-Frolov,'98]*

Getting the PB of Arutyunov-Olivucci (1)

Recall the pencil: $P_{Q_d,\mathtt{mqv}} + \psi_z$, $\quad \psi_z := \sum_{a<b} z_{ab}\, \mathrm{Inf}^{(a)} \wedge \mathrm{Inf}^{(b)}$



(Q_d : plain arrows)

We take $z_{x,v_\alpha} = 0$

$z_{v_\alpha,v_\beta} := z_{\alpha\beta} = \frac{1}{2}$ for $\alpha < \beta$

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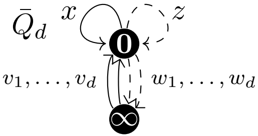

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In terms of the matrices $(X, Z, V_\alpha, W_\alpha)$ on $\mathrm{Rep}(\mathbb{C}\overline{Q}_d, (1,n))$

$$\psi_z(X_{ij}, -) = 0\,, \qquad \psi_z(Z_{ij}, -) = 0\,,$$
$$\psi_z((V_\alpha)_i, (V_\beta)_j) = z_{\alpha\beta}\,(V_\alpha)_i(V_\beta)_j\,, \quad \psi_z((W_\alpha)_i, (W_\beta)_j) = z_{\alpha\beta}\,(W_\alpha)_i(W_\beta)_j\,,$$
$$\psi_z((V_\alpha)_i, (W_\beta)_j) = -z_{\alpha\beta}\,(V_\alpha)_i(W_\beta)_j\,, \quad \psi_z((W_\alpha)_i, (V_\beta)_j) = -z_{\alpha\beta}\,(W_\alpha)_i(V_\beta)_j\,,$$

where $z_{\alpha\alpha=0}$ and $z_{\alpha\beta} := -z_{\beta\alpha}$ if $\alpha > \beta$

Getting the PB of Arutyunov-Olivucci (2)

With the previous choice of parameters $z_{\alpha,\beta}$, we get:

Proposition ([F.,'23])

After writing $P_{Q_d,\mathtt{mqv}} + \psi_z$ in local coordinates, one gets the (minus) PB of [Arutyunov-Olivucci,'20]

$$
\begin{aligned}
\{x_i, x_j\}_- =&0\,, \quad \{x_i, a_j^\alpha\}_- = 0\,, \quad \{x_i, b_j^\alpha\}_- = \delta_{ij} x_i b_j^\alpha\,,\\
\{a_i^\alpha, a_j^\beta\}_- =&\frac{1}{2}\delta_{(i\neq j)}\frac{x_i + x_j}{x_i - x_j}(a_i^\alpha a_j^\beta + a_j^\alpha a_i^\beta - a_j^\alpha a_j^\beta - a_i^\alpha a_i^\beta) + \frac{1}{2} o(\beta,\alpha) a_j^\alpha a_i^\beta\\
&+ \frac{1}{2}\sum_{\gamma=1}^d o(\alpha,\gamma) a_j^\beta a_j^\alpha a_i^\gamma - \frac{1}{2}\sum_{\gamma=1}^d o(\beta,\gamma) a_i^\alpha a_i^\beta a_j^\gamma + \frac{1}{2}\left(\sum_{\gamma,\epsilon} o(\epsilon,\gamma) a_i^\epsilon a_j^\gamma\right) a_i^\alpha a_j^\beta\,,\\
\{a_i^\alpha, b_j^\beta\}_- =&a_i^\alpha Z_{ij} - \delta_{\alpha\beta} Z_{ij} - \frac{1}{2}\delta_{(i\neq j)}\frac{x_i + x_j}{x_i - x_j}(a_i^\alpha - a_j^\alpha) b_j^\beta + a_i^\alpha \sum_{\gamma<\beta} a_i^\gamma b_j^\gamma - \delta_{\alpha\beta}\sum_{\gamma<\beta} a_i^\gamma b_j^\gamma\\
&- \frac{1}{2}\sum_{\gamma=1}^d o(\alpha,\gamma) b_j^\beta a_j^\alpha a_i^\gamma - \frac{1}{2}\left(\sum_{\gamma,\epsilon} o(\epsilon,\gamma) a_i^\epsilon a_j^\gamma\right) a_i^\alpha b_j^\beta + \frac{1}{2} a_i^\alpha a_i^\beta b_j^\beta - \frac{1}{2}\delta_{\alpha\beta} a_i^\alpha b_j^\beta\,,\\
\{b_i^\alpha, b_j^\beta\}_- =&\frac{1}{2}\delta_{(i\neq j)}\frac{x_i + x_j}{x_i - x_j}(b_i^\alpha b_j^\beta + b_j^\alpha b_i^\beta) - b_i^\alpha Z_{ij} + b_j^\beta Z_{ji} - \frac{1}{2} o(\beta,\alpha) b_j^\alpha b_i^\beta\\
&- b_i^\alpha \sum_{\gamma<\beta} a_i^\gamma b_j^\gamma + b_j^\beta \sum_{\gamma<\alpha} a_j^\gamma b_i^\gamma + \frac{1}{2}\left(\sum_{\gamma,\epsilon} o(\epsilon,\gamma) a_i^\epsilon a_j^\gamma\right) b_i^\alpha b_j^\beta + \frac{1}{2}(a_j^\alpha - a_i^\beta) b_i^\alpha b_j^\beta\,,
\end{aligned}
$$

In addition...

As a consequence of the construction:

- large pencil of compatible PB providing many Hamiltonian formulations for the spin RS system
- For each PB in the pencil, we can prove integrability
- This answers positively Conjecture C.2 in [F.-Fehér,'23]

In addition...

As a consequence of the construction:

- large pencil of compatible PB providing many Hamiltonian formulations for the spin RS system
- For each PB in the pencil, we can prove integrability
- This answers positively Conjecture C.2 in [F.-Fehér,'23]

We also obtain:

- "universal" pencil that induces the one on $\mathcal{C}^{\times}_{n,d,q}$,
  i.e. pencil of PB on $\mathrm{Sym}(\mathbb{C}\overline{Q}_d^{loc}/[\mathbb{C}\overline{Q}_d^{loc}, \mathbb{C}\overline{Q}_d^{loc}])$ and surjective Poisson morphism onto any $\mathcal{M}(Q_d, \mathbf{n}, q)$

Open problem:

- Give an alternative construction using Poisson-Lie theory by gathering [Arutyunov-Olivucci,'20] and the mixed products of [Lu-Mouquin,'17]
- Construct such pencils for any CM-RS system

Thank you for your attention !

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