Modified double brackets and a conjecture of Arthamonov

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Plan for the talk

1. **(Modified) double brackets**
2. The approach of Goncharov-Gubarev
3. Mixed double Poisson algebras

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Motivation

Field $\mathbb{k}$ char. 0, $\overline{\mathbb{k}} = \mathbb{k}$

Following [Kontsevich, '93] and [Kontsevich-Rosenberg, '99] $(n \in \mathbb{N}^\times)$

$$\begin{array}{ccc} \text{associative } \mathbb{k}\text{-algebra} & \to & \text{commutative } \mathbb{k}\text{-algebra} \\ A & \longrightarrow & \mathbb{k}[\mathrm{Rep}(A,n)] \end{array}$$

$\mathbb{k}[\mathrm{Rep}(A,n)]$ is generated by symbols a_{ij}, $\forall a \in A$, $1 \leq i,j \leq n$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Goal: Find a property P_{nc} on A that gives the usual property P on $\mathbb{k}[\mathrm{Rep}(A,n)]$ for all $n \in \mathbb{N}^\times$

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Question: What about having a Poisson bracket?

Double brackets

We follow [Van den Bergh, *double Poisson algebras*,'08]

$A/\mathbb{k}$ associative unital finitely generated

Notation: $\otimes = \otimes_{\mathbb{k}}$; $d \in A^{\otimes 2}$ written $d = d' \otimes d''$

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Definition

A *double bracket* on A is a $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ s.t.

1. $\{\!\{a,b\}\!\} = -\tau_{(12)}\ \{\!\{b,a\}\!\}$ (cyclic skewsymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1)\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}(1 \otimes c)$ (outer derivation)
   $\{\!\{bc,a\}\!\} = (1 \otimes b)\{\!\{c,a\}\!\} + \{\!\{b,a\}\!\}(c \otimes 1)$ (inner derivation)

We use $(a' \otimes a'')(b' \otimes b'') = a'b' \otimes a''b''$

Hereafter, (2) are referred to as the "derivation rules"

Double Poisson bracket

From a double bracket $\{\!\{-,-\}\!\}$, define for $a,b,c\in A$

$$\begin{aligned}\{\!\{a,b\otimes c\}\!\}_L &= \{\!\{a,b\}\!\}\otimes c,\\ \{\!\{a,b\otimes c\}\!\}_R &= b\otimes\{\!\{a,c\}\!\},\\ \{\!\{b\otimes c,a\}\!\}_L &= \{\!\{b,a\}\!\}\otimes_1 c,\end{aligned}$$

Notation: $(a\otimes b)\otimes_1 c = a\otimes c\otimes b = c\otimes_1 (a\otimes b)$.

We consider the *double Jacobiator* $\mathbb{D}\mathrm{Jac}: A^{\otimes 3}\to A^{\otimes 3}$ by

$$\mathbb{D}\mathrm{Jac}(a,b,c) = \{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L - \{\!\{b,\{\!\{a,c\}\!\}\}\!\}_R - \{\!\{\{\!\{a,b\}\!\},c\}\!\}_L\ .$$

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Definition

A double bracket $\{\!\{-,-\}\!\}$ is *Poisson* if $\mathbb{D}\mathrm{Jac} = 0$.

DPA : examples

Key property: enough to define $\{\!\{-,-\}\!\}$ on generators

Example

Take $A=\Bbbk[x]$ (or any quotient $\Bbbk[x]/(x^k)$).
$\{\!\{x,x\}\!\}=x\otimes 1-1\otimes x$

Example

Take $A=\Bbbk\langle x,y\rangle$
$\{\!\{x,y\}\!\}=1\otimes 1$, $\{\!\{x,x\}\!\}=0$, $\{\!\{y,y\}\!\}=0$

Relation to KR principle

Theorem (Van den Bergh,'08)

If A has a double bracket $\{\!\{-,-\}\!\}$, then for any $n \geq 1$ $\Bbbk[\mathrm{Rep}(A,n)]$ has an antisymmetric biderivation $\{-,-\}_n$ defined by

$$\{a_{ij}, b_{kl}\}_n = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}\ .$$

If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}_n$ is a Poisson bracket.

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Corollary (Van den Bergh,'08)

For double Poisson bracket $\{\!\{-,-\}\!\}$ on A,

$$\{\mathrm{tr}(a), \mathrm{tr}(b)\}_n = \mathrm{tr}(\mathrm{m} \circ \{\!\{a,b\}\!\})\,, \qquad \mathrm{tr}(c) := \sum_{1 \leq j \leq n} c_{jj}\,,$$

defines a Poisson bracket on invariant subalgebra $\Bbbk[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$.

DPA : examples and representations

Example

Take $A = \mathbb{k}[x]$ (or any quotient $\mathbb{k}[x]/(x^k)$).
$\{\!\{x, x\}\!\} = x \otimes 1 - 1 \otimes x$
Induces PB of $\mathfrak{gl}_n \simeq \mathbb{k}[\mathrm{Rep}(A, n)]$

Example

Take $A = \mathbb{k}\langle x, y\rangle$
$\{\!\{x, y\}\!\} = 1 \otimes 1$, $\{\!\{x, x\}\!\} = 0$, $\{\!\{y, y\}\!\} = 0$
Induces PB of $T^*\mathfrak{gl}_n \simeq \mathbb{k}[\mathrm{Rep}(A, n)]$

$\rightsquigarrow$ Second example extends to any path algebra of quiver !

Weak version of KR principle

associative $\mathbb{k}$-algebra $\rightarrow$ commutative $\mathbb{k}$-algebra

$$A \longrightarrow \mathbb{k}[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$$

Question:
Can we induce a Poisson bracket on $\mathbb{k}[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$ from some structure on A that may not lift to a Poisson bracket on $\mathbb{k}[\mathrm{Rep}(A,n)]$?

Weak version of KR principle

$$\begin{array}{ccc} \text{associative } \Bbbk\text{-algebra} & \to & \text{commutative } \Bbbk\text{-algebra} \\ A & \longrightarrow & \Bbbk[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n} \end{array}$$

Question:
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1. First approach [Crawley-Boevey,'05,'11] : H_0-Poisson structures
   Problem: hard to manipulate in general
2. Second approach [Arthamonov,'15,'17] : modified double brackets

Modified double brackets

Definition ([Arthamonov,'15,'17])

A $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ satisfying the derivation rules is called a *modified double bracket* if the H_0-skewsymmetry rule holds:

$$\{a,b\} + \{b,a\} \in [A,A] \qquad \forall a,b \in A,$$

where $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$.

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where $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$. When it satisfies

$$\{a,\{b,c\}\} - \{b,\{a,c\}\} - \{\{a,b\},c\} = 0\,,$$

we call it a *modified double Poisson bracket*.

Double Poisson brackets *are* modified double Poisson brackets
Converse is far from true !

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Double Poisson brackets *are* modified double Poisson brackets
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Theorem (Arthamonov,'17)

For a modified double Poisson bracket $\{\!\{-,-\}\!\}$ *on* A,
$\{\mathrm{tr}(a),\mathrm{tr}(b)\}_n = \mathrm{tr}(\{a,b\})$ *defines a Poisson bracket on* $\Bbbk[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$.

The conjecture...

Conjecture ([Arthamonov,'17])

On $A = \Bbbk\langle x_1, x_2, x_3\rangle$, the following operations define modified double Poisson brackets:

$$\begin{aligned}
\{\!\{x_1, x_2\}\!\}^I &= -x_2x_1 \otimes 1, & \{\!\{x_2, x_1\}\!\}^I &= x_1x_2 \otimes 1,\\
\{\!\{x_2, x_3\}\!\}^I &= -x_2 \otimes x_3, & \{\!\{x_3, x_2\}\!\}^I &= x_2 \otimes x_3,\\
\{\!\{x_3, x_1\}\!\}^I &= -1 \otimes x_3x_1, & \{\!\{x_1, x_3\}\!\}^I &= 1 \otimes x_1x_3,
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
\{\!\{x_1, x_2\}\!\}^{II} &= -x_1 \otimes x_2, & \{\!\{x_2, x_1\}\!\}^{II} &= x_1 \otimes x_2,\\
\{\!\{x_2, x_3\}\!\}^{II} &= x_3 \otimes x_2, & \{\!\{x_3, x_2\}\!\}^{II} &= -x_3 \otimes x_2,\\
\{\!\{x_3, x_1\}\!\}^{II} &= x_1 \otimes x_3 - x_3 \otimes x_1, & &
\end{aligned} \tag{2}$$

where the remaining omitted terms involving pairs of generators are assumed to be zero.

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2. **The approach of Goncharov-Gubarev**
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Observation

$$
\begin{aligned}
\{\!\!\{x_1, x_2\}\!\!\}^{II} &= -x_1 \otimes x_2, & \{\!\!\{x_2, x_1\}\!\!\}^{II} &= x_1 \otimes x_2, \\
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\{\!\!\{x_3, x_1\}\!\!\}^{II} &= x_1 \otimes x_3 - x_3 \otimes x_1, & &
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The map $\{\!\!\{-,-\}\!\!\}^{II}$ restricts to $V \times V \to V \otimes V$ for $V = \oplus_{j=1}^{3} \Bbbk x_j$

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Definition ([Schedler,'09;ORS,'13;DSKV,'15])

A $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : V \times V \to V \otimes V$ for vector space V satisfying Van den Bergh's cyclic skewsymmetry and $\mathbb{D}\mathrm{Jac} = 0$ turns V into a *double Lie algebra*.

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Problem: $\{\!\{-,-\}\!\}^{II}$ is *not* a double Lie algebra

$$\rightsquigarrow \quad \{\!\{x_1, x_2\}\!\}^{II} + \tau_{(12)} \{\!\{x_2, x_1\}\!\}^{II} = -(x_1 \otimes x_2 - x_2 \otimes x_1)$$

What to do ???

Another observation

Fix a double Lie algebra $(V, \{\!\{-,-\}\!\})$.
Fix dual bases (e_k) and (e^k) of $\mathrm{End}(V) \simeq \mathrm{Mat}_d(\Bbbk)$ under trace pairing

We can define $R : \mathrm{End}(V) \to \mathrm{End}(V)$ uniquely through

$$\{\!\{u, v\}\!\} = \sum_{1 \leq k \leq d^2} e_k(u) \otimes R(e^k)(v)\,, \quad u, v \in V\,.$$

Lemma ([Goncharov-Kolesnikov,'18])

R is a skewsymmetric Rota-Baxter operator on $\mathrm{End}(V)$, i.e.

1. $R = -R^*$ *(dual for trace pairing);*
2. $R(e)R(f) = R(R(e)f + eR(f))$, $\forall e, f \in \mathrm{End}(V)$.

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Side note : equivalent to skewsymmetric solution of AYBE on $\mathrm{End}(V)$,
i.e. $r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0$ and $r + \tau_{(12)}r = 0$ for $r \in \mathrm{End}(V)^{\otimes 2}$

λ-double Lie algebras

Idea of [Goncharov-Gubarev,'22] : consider RB operator of weight $\lambda \in \mathbb{k}^{\times}$:

$$R(e)R(f) = R(R(e)f + eR(f) + \lambda\, ef), \qquad \forall e, f \in \mathrm{End}(V)$$

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A $\Bbbk$-bilinear map $\{\!\{-,-\}\!\} : V \times V \to V \otimes V$ for vector space V satisfying

- $\{\!\{u,v\}\!\} + \tau_{(12)} \{\!\{v,u\}\!\} = \lambda(u \otimes v - v \otimes u)$ for $u, v \in V$;
- $\mathbb{D}\mathrm{Jac}(u,v,w) = -\lambda\, v \otimes_1 \{\!\{u,w\}\!\}$ for $u, v, w \in V$;

turns V into a *λ-double Lie algebra*.

Conjecture for $\{\!\{-,-\}\!\}^{II}$

Theorem ([GG,'22])

A λ-double Lie algebra structure on a vector space V extends uniquely through the derivation rules to a modified double Poisson bracket on $A = \mathrm{Ass}(V)$.
In particular, this induces a Poisson bracket on $\Bbbk[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$.

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Proposition ([GG,'22])

Arthamonov's map $\{\!\{-,-\}\!\}^{II}$ is a (-1)-double Lie algebra on the vector space $V = \oplus_{j=1}^{3} \Bbbk x_j$.

Corollary ([GG,'22])

$\{\!\{-,-\}\!\}^{II}$ defines a modified double Poisson bracket on $A = \Bbbk\langle x_1, x_2, x_3\rangle$.

$\rightsquigarrow$ This proves half of the conjecture

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The problem of $\{\!\{-,-\}\!\}^I$

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Problem 1 : does not restrict to $V\times V\to V\otimes V$ for $V=\oplus_{j=1}^3 \Bbbk x_j$

Problem 2 : not $\{\!\{u,v\}\!\}+\tau_{(12)}\{\!\{v,u\}\!\}=\lambda(u\otimes v-v\otimes u)$ as

$$\{\!\{x_1,x_2\}\!\}^I+\tau_{(12)}\{\!\{x_2,x_1\}\!\}^I = -x_2x_1\otimes 1+1\otimes x_1x_2,$$
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$$\{\!\{x_3,x_1\}\!\}^I + \tau_{(12)}\{\!\{x_1,x_3\}\!\}^I = -1\otimes x_3x_1 + x_1x_3\otimes 1$$

Idea: Should consider something like : for $u,v\in V$

$$\{\!\{u,v\}\!\}+\tau_{(12)}\{\!\{v,u\}\!\} = \lambda(u,v)(u\otimes v - v\otimes u) + \mu(u,v)(1\otimes uv - vu\otimes 1)$$

First step

For $d \geq 1$, let $A = \Bbbk\langle v_1, \ldots, v_d \rangle$ We follow [F. '25; arXiv:2405.17930]

Lemma

Given a bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ satisfying the derivation rules and for any $1 \leq i, j \leq d$,

$$\{\!\{v_i, v_j\}\!\} + \{\!\{v_j, v_i\}\!\}^\circ = \underbrace{\lambda_{ij}}_{sym.} (v_i \otimes v_j - v_j \otimes v_i) + \underbrace{\mu_{ij}}_{skewsym.} (1 \otimes v_i v_j - v_j v_i \otimes 1)$$

then it satisfies Arthamonov's H_0-skewsymmetry when

$$\lambda_{ij} - \lambda_{kl} = \mu_{il} - \mu_{kj} \text{ for all } 1 \leq i, j, k, l \leq d.$$

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Proof.

For $a = v_{i_1} \cdots v_{i_r}$ and $b = v_{j_1} \cdots v_{j_s}$, we have modulo $[A, A]$

$$\mathrm{m} \circ (\{\!\{a, b\}\!\} + \{\!\{b, a\}\!\}^\circ) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{s} (\lambda_{i_\alpha j_\beta} - \lambda_{i_{\alpha-1} j_{\beta-1}} + \mu_{i_{\alpha-1} j_\beta} - \mu_{i_\alpha j_{\beta-1}}) \mathrm{V}_{\alpha,\beta}$$

for $\mathrm{V}_{\alpha,\beta} := v_{i_\alpha} a_\alpha^+ a_\alpha^- v_{j_\beta} b_\beta^+ b_\beta^-$. □

First definition

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Definition

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{k}^d$. A bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ satisfying the derivation rules and for all $1 \leq i, j \leq d$

$$\{\!\{v_i, v_j\}\!\} + \{\!\{v_j, v_i\}\!\}^\circ = \frac{\lambda_i + \lambda_j}{2}(v_i \otimes v_j - v_j \otimes v_i) + \frac{\lambda_i - \lambda_j}{2}(1 \otimes v_i v_j - v_j v_i \otimes 1)$$

turns the pair $(A, \{\!\{-,-\}\!\})$ into a *mixed double algebra of weight* $\underline{\lambda}$.

Corollary

A mixed double algebra of weight $\underline{\lambda} \in \mathbb{k}^d$ *is equipped with a modified double bracket.*

Main definition

Let again $A = \mathbb{k}\langle v_1, \dots, v_d\rangle$.

Definition

A mixed double algebra $(A, \{\!\{-,-\}\!\})$ of weight $\underline{\lambda} \in \mathbb{k}^d$ is *Poisson* when, for any $1 \leq i, j, k \leq d$,

$$\mathbb{D}\mathrm{Jac}(v_i, v_j, v_k) = -\frac{\lambda_i + \lambda_j}{2} v_j \otimes_1 \{\!\{v_i, v_k\}\!\} + \frac{\lambda_i - \lambda_j}{2} 1 \otimes_1 ((1 \otimes v_j) \{\!\{v_i, v_k\}\!\}).$$

First term means $\{\!\{v_i, v_k\}\!\}' \otimes v_j \otimes \{\!\{v_i, v_k\}\!\}''$
Second term means $\{\!\{v_i, v_k\}\!\}' \otimes 1 \otimes v_j \{\!\{v_i, v_k\}\!\}''$

Main definition

Let again $A = \Bbbk\langle v_1, \dots, v_d \rangle$.

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Remark: If all $\lambda_j = \lambda$, these are the conditions of [Goncharov-Gubarev,'22]

Main result

Proposition

Fix $(A, \{\!\{-,-\}\!\})$ *a mixed double Poisson algebra of weight* $\underline{\lambda} \in \mathbb{k}^d$*.*
Then the Jacobi identity

$$\{a,\{b,c\}\} - \{b,\{a,c\}\} - \{\{a,b\},c\} = 0\,,$$

is satisfied for any $a, b, c \in A$*.*

Main result

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Fix $(A, \{\!\{-,-\}\!\})$ *a mixed double Poisson algebra of weight* $\underline{\lambda} \in \mathbb{k}^d$*. Then the Jacobi identity*

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is satisfied for any $a, b, c \in A$*.*

Theorem

If $(A, \{\!\{-,-\}\!\})$ *is a mixed double Poisson algebra of weight* $\underline{\lambda}$*, then* $\{\!\{-,-\}\!\}$ *is a modified double Poisson bracket.*

Conjecture for $\{\!\{-,-\}\!\}^I$

Theorem

$\{\!\{-,-\}\!\}^I$ defines a mixed double Poisson algebra on $A = \mathbb{k}\langle x_1, x_2, x_3\rangle$ of weight $(-1,-1,1)$.

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Corollary

$\{\!\{-,-\}\!\}^I$ defines a modified double Poisson bracket on $A = \Bbbk\langle x_1, x_2, x_3\rangle$.

$\rightsquigarrow$ This proves* the second half of the conjecture

*Independent proof of the conjecture by V.Gubarev's student A.Savel'ev at Novosibirsk State U.

Some side results (1)

Proposition

The algebra $\Bbbk\langle v, w\rangle$ *is a mixed double Poisson algebra of weight* $(1,-1)$ *if it is equipped with the operation* $\{\!\{-,-\}\!\}$ *satisfying one of the following four conditions*

$$\begin{aligned} \{\!\{v,w\}\!\} &= 0, & \{\!\{w,v\}\!\} &= -1\otimes wv + vw\otimes 1;\\ \{\!\{v,w\}\!\} &= 1\otimes vw, & \{\!\{w,v\}\!\} &= -1\otimes wv;\\ \{\!\{v,w\}\!\} &= -wv\otimes 1, & \{\!\{w,v\}\!\} &= vw\otimes 1;\\ \{\!\{v,w\}\!\} &= 1\otimes vw - wv\otimes 1, & \{\!\{w,v\}\!\} &= 0. \end{aligned}$$

Proposition

$\Bbbk\langle v_1, v_2, v_3\rangle$ *is a mixed double algebra of weight* $(1,1,-1)$ *for*

$$\{\!\{v_1,v_1\}\!\} = 0,\quad \{\!\{v_2,v_2\}\!\} = 0,\quad \{\!\{v_3,v_3\}\!\} = 0,$$

$$\{\!\{v_1,v_2\}\!\} = \tilde\alpha_3\, v_1\otimes v_2 - \tilde\beta_3\, v_2\otimes v_1,\quad \{\!\{v_2,v_1\}\!\} = (-1+\tilde\beta_3)\, v_1\otimes v_2 + (1-\tilde\alpha_3)\, v_2\otimes v_1\,,$$

$$\{\!\{v_1,v_3\}\!\} = \alpha_2\, 1\otimes v_1v_3 - \beta_2\, v_3v_1\otimes 1,\quad \{\!\{v_3,v_1\}\!\} = (-1+\beta_2)\, 1\otimes v_3v_1 + (1-\alpha_2)\, v_1v_3\otimes 1\,,$$

$$\{\!\{v_2,v_3\}\!\} = \alpha_1\, 1\otimes v_2v_3 - \beta_1\, v_3v_2\otimes 1,\quad \{\!\{v_3,v_2\}\!\} = (-1+\beta_1)\, 1\otimes v_3v_2 + (1-\alpha_1)\, v_2v_3\otimes 1\,.$$

with $\alpha_1,\alpha_2,\tilde\alpha_3,\beta_1,\beta_2,\tilde\beta_3\in\Bbbk$*. Furthermore,* $\{\!\{-,-\}\!\}$ *is Poisson when the triples* $(\alpha_1,\alpha_2,\tilde\beta_3)$ *and* $(\beta_1,\beta_2,\tilde{\alpha_3})$ *take one of the following* 6 *values*

$$(0,0,0),\ (1,0,0),\ (0,0,1),\ (1,1,0),\ (0,1,1),\ (1,1,1).$$

Some side results (2)

Proposition

Fix $d \geq 3$ and $0 \leq \delta \leq d$. The following defines a mixed double Poisson algebra structure on $\Bbbk\langle v_1, \ldots, v_d\rangle$ of weight $\mathbf{1}_{\delta,d}$:

$$\begin{aligned}
&\{\!\!\{v_i, v_i\}\!\!\} = 0, \quad \text{for } 1 \leq i \leq d,\\
&\{\!\!\{v_i, v_j\}\!\!\} = v_i \otimes v_j - v_j \otimes v_i, \quad \{\!\!\{v_j, v_i\}\!\!\} = 0\,, \quad \text{for } 1 \leq i < j \leq \delta\,,\\
&\{\!\!\{v_i, v_k\}\!\!\} = 1 \otimes v_i v_k - v_k v_i \otimes 1, \quad \{\!\!\{v_k, v_i\}\!\!\} = 0\,, \quad \text{for } 1 \leq i \leq \delta < k \leq d\,,\\
&\{\!\!\{v_k, v_l\}\!\!\} = -v_k \otimes v_l + v_l \otimes v_k, \quad \{\!\!\{v_l, v_k\}\!\!\} = 0\,, \quad \text{for } \delta < k < l \leq d\,.
\end{aligned}$$

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&\{\!\!\{v_i, v_k\}\!\!\} = -v_k v_i \otimes 1, \quad \{\!\!\{v_k, v_i\}\!\!\} = v_i v_k \otimes 1\,, \quad \text{for } 1 \leq i \leq \delta < k \leq d\,,\\
&\{\!\!\{v_k, v_l\}\!\!\} = -v_k \otimes v_l, \quad \{\!\!\{v_l, v_k\}\!\!\} = v_k \otimes v_l\,, \quad \text{for } \delta < k < l \leq d\,.
\end{aligned}$$

Some problems

• Does there exist a mixed double Poisson algebra structure on A_2 of weight (λ_1, λ_2) with $\lambda_1 \neq \pm\lambda_2$?
• Non-zero self-brackets? ($\{\!\{v_i, v_i\}\!\} \neq 0$ for v_i generator)
• For any $\underline{\lambda} \in \mathbb{k}^d$, does there exist a mixed double Poisson algebra structure on A_d of that weight?

• Find examples of mixed double Poisson algebras that are not simply obtained by localisation/quotient of such a structure on a free algebra.

• Reformulate the conditions red and blue with operator R (or a pair?)
$\rightsquigarrow$ Generalize R λ-skew-symmetric RB operator of weight λ

Thank you for your attention !

Details and references in [arXiv:2405.17930]

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