Double Poisson algebra cohomology

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Plan for the talk

1. **Motivation and classical constructions**
2. NC Poisson geometry and a first cohomology theory
3. Completed double Poisson algebra cohomology
4. Final remarks

Based on arXiv:2509.21232 with Daniele Valeri (La Sapienza, IT)

Classical Poisson cohomology (1)

Fix a *commutative associative* $\mathbb{k}$-algebra A

Recall $(A, \{-,-\})$ is a Poisson algebra if $\{-,-\} : A \otimes A \to A$ linear s.t.

(skewsymmetry) $\{x,y\} = -\{y,x\}$

(Leibniz rules) $\{x,yz\} = y\{x,z\} + \{x,y\}z\,,\quad \{xy,z\} = x\{y,z\} + \{x,z\}y$

(Jacobi identity) $\{x,\{y,z\}\} - \{y,\{x,z\}\} = \{\{x,y\},z\}$

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$\mathfrak{X}^n(A) \subset \mathrm{Hom}(\wedge^n A, A)$ spanned by elements P s.t.
$P(ab, a_2, \dots, a_n) = aP(b, a_2, \dots, a_n) + bP(a, a_2, \dots, a_n)\,.$

$$\delta^n_{CE}(P)(x_1, \dots, x_{n+1}) = \sum_{1 \le i \le n+1} (-1)^{i+1}\{x_i, P(x_1, \overset{i}{\check{\dots}}, x_{n+1})\}$$
$$+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} P(\{x_i, x_j\}, x_1, \overset{i}{\check{\dots}}, \overset{j}{\check{\dots}}, x_{n+1})\,.$$

From CE differential $\rightsquigarrow \quad \mathrm{H}_{CE}(A) := \mathrm{H}(\mathfrak{X}(A), \delta_{CE})$.

Classical Poisson cohomology (2)

Schouten-Nijenhuis bracket $[-,-]_{\mathrm{SN}}$ on $\mathfrak{X}^k(A)$ $\quad (P \in \mathfrak{X}^k(A),\ Q \in \mathfrak{X}^l(A))$

$$\begin{aligned}
&[P,Q]_{\mathrm{SN}}(a_1,\ldots,a_{k+l-1})\\
=&(-1)^{(k-1)(l-1)} \sum_{\sigma \in S_{l,k-1}} \mathrm{sgn}(\sigma) P(Q(a_{\sigma(1)},\ldots,a_{\sigma(l)}), a_{\sigma(l+1)},\ldots,a_{\sigma(k+l-1)})\\
&- \sum_{\sigma \in S_{k,l-1}} \mathrm{sgn}(\sigma) Q(P(a_{\sigma(1)},\ldots,a_{\sigma(k)}), a_{\sigma(k+1)},\ldots,a_{\sigma(k+l-1)}),
\end{aligned}$$

$\rightsquigarrow$ $(\mathfrak{X}(A), [-,-]_{\mathrm{SN}})$ is a graded Lie algebra of degree -1, in particular
$[x,[y,z]_{\mathrm{SN}}]_{\mathrm{SN}} - (-1)^{(|x|-1)(|y|-1)}[y,[x,z]_{\mathrm{SN}}]_{\mathrm{SN}} = [[x,y]_{\mathrm{SN}},z]_{\mathrm{SN}}$

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$$\boxed{\mathrm{d}_\Pi := [\Pi,-]_{\mathrm{SN}}, \qquad \Pi \in \mathfrak{X}^2(A) \text{ s.t. } [\Pi,\Pi]_{\mathrm{SN}} = 0}$$

$\rightsquigarrow$ $\mathrm{PH}(A) := \mathrm{H}(\mathfrak{X}(A), \mathrm{d}_\Pi)$

For $\{-,-\}_\Pi : A \times A \to A$, $\{a,b\}_\Pi := \Pi(a,b)$, get $\mathrm{d}_\Pi = \delta_{CE}$
$\rightsquigarrow$ $\mathrm{H}_{CE}(A) \simeq \mathrm{PH}(A)$

Kontsevich-Rosenberg principle

Field $\Bbbk$ char. 0, $\overline{\Bbbk} = \Bbbk$

Following [Kontsevich, '93] and [Kontsevich-Rosenberg, '99] $(N \in \mathbb{N}^\times)$

$$\begin{array}{ccc} \text{associative } \Bbbk\text{-algebra} & \to & \text{commutative } \Bbbk\text{-algebra} \\ \mathcal{A} & \longrightarrow & \mathcal{A}_N := \Bbbk[\mathrm{Rep}(\mathcal{A}, N)] \end{array}$$

$\mathcal{A}_N$ is generated by symbols a_{ij}, $\forall a \in \mathcal{A}$, $1 \leq i, j \leq N$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Goal: Find a property P_{nc} on $\mathcal{A}$ that gives the usual property P on $\mathcal{A}_N$ for all $N \in \mathbb{N}^\times$

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Question: Can we induce a Poisson bracket and its Poisson cohomology?

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Double (Poisson) brackets

$\mathcal{A}$ is a unital associative fin. gen. algebra over $\Bbbk$, $\otimes := \otimes_{\Bbbk}$

Definition ([Van den Bergh,'08])

A double bracket is a $\Bbbk$-linear map $\{\!\{-,-\}\!\} : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ s.t.

1. $\{\!\{a,b\}\!\} = -\tau_{(12)}\,\{\!\{b,a\}\!\}$ (cyclic skewsymmetry)
2. $\{\!\{a,bc\}\!\} = (b\otimes 1)\,\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}\,(1\otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1\otimes a)\,\{\!\{d,b\}\!\} + \{\!\{a,b\}\!\}\,(d\otimes 1)$ (inner derivation)

A double bracket is Poisson if "double Jacobiator" $\mathbb{D}\mathrm{Jac} : \mathcal{A}^{\otimes 3} \to \mathcal{A}^{\otimes 3}$,

$$\mathbb{D}\mathrm{Jac}(a,b,c) = \{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L + \tau_{(123)}\,\{\!\{b,\{\!\{c,a\}\!\}\}\!\}_L + \tau_{(132)}\,\{\!\{c,\{\!\{a,b\}\!\}\}\!\}_L$$

is identically zero.

In the def, we use $\{\!\{a, X'\otimes X''\}\!\}_L := \{\!\{a,X'\}\!\}\otimes X''$

More generally, B-linear version that I will omit

DPA : examples

Key property: enough to define $\{\!\{-,-\}\!\}$ on generators

Example

Take $\mathbb{k}[x]$ (or any quotient $\mathbb{k}[x]/(x^k)$).
$\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$

Example

Take $\mathbb{k}\langle x,y\rangle$
$\{\!\{x,y\}\!\} = 1 \otimes 1$, $\{\!\{x,x\}\!\} = 0$, $\{\!\{y,y\}\!\} = 0$

Relation to the Kontsevich-Rosenberg principle

$\mathcal{A}_N$ generated by symbols a_{ij}, $\forall a \in \mathcal{A}$, $1 \leq i, j \leq N$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Theorem (Van den Bergh,'08)

If $\mathcal{A}$ has a double bracket $\{\!\{-,-\}\!\}$, then for any $N \geq 1$ $\mathcal{A}_N := \Bbbk[\mathrm{Rep}(\mathcal{A}, N)]$ has an antisymmetric biderivation $\{-,-\}$ defined by

$$\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj} \ \{\!\{a,b\}\!\}''_{il} \ .$$

If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}$ is a Poisson bracket.

Proof.

$\mathrm{Jac}(a_{ij}, b_{kl}, c_{uv}) = \mathbb{D}\mathrm{Jac}(a,b,c)_{uj,il,kv} - \mathbb{D}\mathrm{Jac}(a,c,b)_{kj,iv,ul}$ $\square$

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$\Rightarrow$ Set $\mathrm{tr}(c) := \sum_j c_{jj}$. Then $\{\mathrm{tr}(a), \mathrm{tr}(b)\} = \mathrm{tr}(\mathrm{m} \circ \{\!\{a,b\}\!\})$ defines a Poisson bracket on the invariant subalgebra $\mathcal{A}_N^{\mathrm{GL}_N}$.

DPA : examples and representations

Example

Take $\Bbbk[x]$ (*or any quotient* $\Bbbk[x]/(x^r)$)

$\{\!\{x, x\}\!\} = x \otimes 1 - 1 \otimes x$

Induces PB of $\mathfrak{gl}_N \simeq \Bbbk[\mathrm{Rep}(\Bbbk[x], N)]$ (*or nilpotent matrices of order* r)

Example

Take $\Bbbk\langle x, y\rangle$

$\{\!\{x, y\}\!\} = 1 \otimes 1$, $\{\!\{x, x\}\!\} = 0$, $\{\!\{y, y\}\!\} = 0$

Induces PB of $T^*\mathfrak{gl}_N \simeq \Bbbk[\mathrm{Rep}(\Bbbk\langle x, y\rangle, N)]$

$\rightsquigarrow$ Second example extends to any path algebra of quiver !

multibrackets and NC forms

A n-bracket on $\mathcal{A}$ is a linear map $\{\!\{-,\dots,-\}\!\} : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$ such that (cyclic skewsymmetry) $+$ (Leibniz rule), i.e.

$$\{\!\{a_1, a_2, \dots, a_n\}\!\} = (-1)^{n-1} \{\!\{a_2, \dots, a_n, a_1\}\!\}^{\sigma},$$

$$\{\!\{a_1, \dots, a_{n-1}, bc\}\!\} = b\,\{\!\{a_1, \dots, a_{n-1}, c\}\!\} + \{\!\{a_1, \dots, a_{n-1}, b\}\!\}\, c\,.$$

$\mathrm{BR}(\mathcal{A})_n = \mathrm{span}_{\Bbbk}\{$all n-brackets$\}$, $\mathrm{BR}(\mathcal{A}) := \oplus_{n\geq 1} \mathrm{BR}(\mathcal{A})_n$

Completed in degree 0 : $\widehat{\mathrm{BR}}(\mathcal{A}) = \oplus_{n\geq 0}\widehat{\mathrm{BR}}(\mathcal{A})_n$ with $\widehat{\mathrm{BR}}(\mathcal{A})_0 = \mathcal{A}_\sharp := \mathcal{A}/[\mathcal{A},\mathcal{A}]$ and $\widehat{\mathrm{BR}}(\mathcal{A})_n = \mathrm{BR}(\mathcal{A})_n$

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$\mathcal{A}$-bimodule of double derivations

$$\mathbb{D}\mathrm{er}(\mathcal{A}) := \{D \in \mathrm{Hom}_{\Bbbk}(\mathcal{A}, \mathcal{A}^{\otimes 2}) \mid D(ab) = a\, D(b) + D(a)\, b\}$$

Noncommutative multivector fields: $\mathbb{T}^*\mathcal{A} := T_{\mathcal{A}}\mathbb{D}\mathrm{er}(\mathcal{A})$

Differential DPA

Proposition (Van den Bergh,'08)

For $n \geq 1$, there is a well-defined map $\mu_n : (\mathbb{T}^*\mathcal{A})_n \longrightarrow \widehat{\mathrm{BR}}(\mathcal{A})_n$,

$$\mu_n(Q) := \{\!\{-,\dots,-\}\!\}_Q = \textstyle\sum_{0\leq i\leq n-1}(-1)^{(n-1)i}\sigma^i \circ \{\!\{-,\dots,-\}\!\}_Q^{\sim} \circ \sigma^{-i}$$

obtained $\Bbbk$-linearly from ($a_j \in \mathcal{A}$ and $\delta_j \in \mathbb{D}er(\mathcal{A})$)

$$\{\!\{a_1,\dots,a_n\}\!\}^{\sim}_{\delta_1\dots\delta_n} := \delta_n(a_n)'\delta_1(a_1)'' \otimes \delta_1(a_1)'\delta_2(a_2)'' \otimes \cdots \otimes \delta_{n-1}(a_{n-1})'\delta_n(a_n)''$$

Moreover, μ_n factors through $(\mathbb{T}^*\mathcal{A})_{\sharp,n}$, i.e. $\mu_n(Q)$ only depends on Q modulo graded commutators.

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Moreover, μ_n factors through $(\mathbb{T}^*\mathcal{A})_{\natural,n}$, i.e. $\mu_n(Q)$ only depends on Q modulo graded commutators.

Under some assumptions, all μ_n are isomorphisms [Van den Bergh,'08]

A double bracket $\{\!\{-,-\}\!\}$ is differential if $\{\!\{-,-\}\!\} = \mu_2(P)$, $P \in (\mathbb{T}^*\mathcal{A})_2$

Example

$\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$ is differential on $\mathbb{k}[x]$, but not on $\mathbb{k}[x]/(x^k)$, $k \geq 2$.

What we are going to see...

Recall for the standard case:

- $\mathrm{H}_{CE}(A) = \mathrm{H}(\mathfrak{X}(A), \delta_{CE})$ defined from $\{-,-\}$
- $\mathrm{PH}(A) = \mathrm{H}(\mathfrak{X}(A), \mathrm{d}_\Pi = [\Pi, -]_{\mathrm{SN}})$ defined from bivector Π
- One has $\mathrm{H}_{CE}(A) \simeq \mathrm{PH}(A)$

At the NC level:

- $\widehat{\mathrm{dPH}}(\mathcal{A})$ defined on $\widehat{\mathrm{BR}}(\mathcal{A})$ from a double Poisson bracket (this is the new approach of [F.-Valeri,2509.21232])
- $\mathrm{dPH}(\mathcal{A})$ defined on $(\mathbb{T}^*\mathcal{A})_\sharp$, $\mathrm{d}_P = \{P, -\}_{\mathrm{SN}}$ (approach of [Pichereau-Van de Weyer,'08; math/0701837])
- $(\mu_n)_n$ induce a linear map $\mathrm{dPH}(\mathcal{A}) \to \widehat{\mathrm{dPH}}(\mathcal{A})$ if the double Poisson bracket is $\mu_2(P)$

See also [Chemla,'20; 1712.05619] for double Lie-Rinehart algebra cohomologies and [Zheng-Tan, private comm.] for a version of $\{-,-\}_{\mathrm{SN}}$ on $\widehat{\mathrm{BR}}(\mathcal{A})$

NC analogue of SN bracket

Theorem (Van den Bergh,'08)

There is a unique graded double Poisson bracket $\{\!\{-,-\}\!\}_{\mathrm{SN}}$ *of degree* -1 *on* $\mathbb{T}^*\mathcal{A}$ *determined by (*$a, b \in \mathcal{A}$ *and* $\delta, \Delta \in \mathbb{D}\mathrm{er}(\mathcal{A})$*)*

$$\begin{aligned}
\{\!\{a, b\}\!\}_{\mathrm{SN}} &= 0\,, \qquad \{\!\{\delta, a\}\!\}_{\mathrm{SN}} = \delta(a)\,, \\
\{\!\{\delta, \Delta\}\!\}_{\mathrm{SN}} &= \tau_{(23)}\Big((\delta \otimes \mathrm{Id}_{\mathcal{A}}) \circ \Delta - (\mathrm{Id}_{\mathcal{A}} \otimes \Delta) \circ \delta\Big) \\
&\quad + \tau_{(12)}\Big((\mathrm{Id}_{\mathcal{A}} \otimes \delta) \circ \Delta - (\Delta \otimes \mathrm{Id}_{\mathcal{A}}) \circ \delta\Big)\,.
\end{aligned}$$

$\rightsquigarrow$ $(\mathbb{T}^*\mathcal{A})_\natural$ *inherits a graded Lie bracket* $\{-,-\}_{\mathrm{SN}} := \mathrm{m} \circ \{\!\{-,-\}\!\}_{\mathrm{SN}}$.

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A differential double bracket $\mu_2(P)$ is Poisson **<u>iff</u>** $\{P,P\}_{\mathrm{SN}}=0$ in $(\mathbb{T}^*\mathcal{A})_\sharp$

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$\rightsquigarrow$ $(\mathbb{T}^*\mathcal{A})_\natural$ *inherits a graded Lie bracket* $\{-,-\}_{\mathrm{SN}} := \mathrm{m} \circ \{\!\{-,-\}\!\}_{\mathrm{SN}}$.

A differential double bracket $\mu_2(P)$ is Poisson **iff** $\{P, P\}_{\mathrm{SN}} = 0$ in $(\mathbb{T}^*\mathcal{A})_\natural$

$\rightsquigarrow$ $\mathrm{d}_P := \{P, -\}_{\mathrm{SN}}$ is a square-zero differential on $(\mathbb{T}^*\mathcal{A})_\natural$ if $\{P, P\}_{\mathrm{SN}} = 0$

$\rightsquigarrow$ Double Poisson cohomology: $\mathrm{dPH}(\mathcal{A}) = \mathrm{H}((\mathbb{T}^*\mathcal{A})_\natural, \mathrm{d}_P)$

[Van de Weyer,'08], [Pichereau-Van de Weyer,'08]

From dPA cohomology to PA cohomology

$\rightsquigarrow$ $\mathrm{d}_P := \{P,-\}_{\mathrm{SN}}$ is a square-zero differential on $(\mathbb{T}^*\mathcal{A})_\sharp$ if $\{P,P\}_{\mathrm{SN}} = 0$

$\rightsquigarrow$ Double Poisson cohomology: $\mathrm{dPH}(\mathcal{A}) = \mathrm{H}((\mathbb{T}^*\mathcal{A})_\sharp, \mathrm{d}_P)$

Theorem (Pichereau-Van de Weyer,'08)

If $P \in (\mathbb{T}^*\mathcal{A})_{\sharp,2}$ *satisfies* $\{P,P\}_{\mathrm{SN}} = 0$*, there is a morphism of complexes*

$$\mathrm{tr} : ((\mathbb{T}^*\mathcal{A})_\sharp, \mathrm{d}_P) \longrightarrow (\mathfrak{X}(\mathcal{A}_N), \mathrm{d}_{\mathrm{tr}(P)}),\ Q \mapsto \mathrm{tr}(Q),$$

which descends to a linear map $\mathrm{dPH}(\mathcal{A}) \to \mathrm{PH}(\mathcal{A}_N)$ *in cohomology.*
Furthermore, by $\mathrm{GL}_{\mathbf{n}}$*-invariance, get a linear map* $\mathrm{dPH}(\mathcal{A}) \to \mathrm{PH}(\mathcal{A}_{\mathbf{n}}^{\mathrm{GL}_{\mathbf{n}}})$.

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{T}^*\mathcal{A})_{\sharp,0} \arrow[r, "\mathrm{d}_P"'] \arrow[d, "\mathrm{tr}"'] & (\mathbb{T}^*\mathcal{A})_{\sharp,1} \arrow[r, "\mathrm{d}_P"'] \arrow[d, "\mathrm{tr}"'] & (\mathbb{T}^*\mathcal{A})_{\sharp,2} \arrow[r, "\mathrm{d}_P"'] \arrow[d, "\mathrm{tr}"'] & (\mathbb{T}^*\mathcal{A})_{\sharp,3} \arrow[r] \arrow[d, "\mathrm{tr}"'] & \cdots \\
0 \arrow[r] & \mathcal{A}_{\mathbf{n}}^{\mathrm{GL}_{\mathbf{n}}} \arrow[r, "\mathrm{d}_{\mathrm{tr}(P)}"] \arrow[d] & \mathfrak{X}^1(\mathcal{A}_{\mathbf{n}})^{\mathrm{GL}_{\mathbf{n}}} \arrow[r, "\mathrm{d}_{\mathrm{tr}(P)}"] \arrow[d] & \mathfrak{X}^2(\mathcal{A}_{\mathbf{n}})^{\mathrm{GL}_{\mathbf{n}}} \arrow[r, "\mathrm{d}_{\mathrm{tr}(P)}"] \arrow[d] & \mathfrak{X}^3(\mathcal{A}_{\mathbf{n}})^{\mathrm{GL}_{\mathbf{n}}} \arrow[r] \arrow[d] & \cdots \\
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\end{tikzcd}$$

What else?

In [Van de Weyer,'08], [Pichereau-Van de Weyer,'08], computation of several cohomology groups.

BUT

- calculations are tedious;
- there is no cohomology theory if the double Poisson bracket is not differential!
- one misses a "Chevalley-Eilenberg" version of this construction.

Plan for the talk

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The differential

We follow Chapter 4 of [F-Valeri,2509.21232]

Recall: $\widehat{\mathrm{BR}}(\mathcal{A}) = \oplus_{n\geq 0}\widehat{\mathrm{BR}}(\mathcal{A})_n$ with
$\widehat{\mathrm{BR}}(\mathcal{A})_0 = \mathcal{A}_\sharp := \mathcal{A}/[\mathcal{A},\mathcal{A}]$ and $\widehat{\mathrm{BR}}(\mathcal{A})_n = \mathrm{span}_{\Bbbk}\{$all n-brackets$\}$

We denote the double Poisson bracket as $[\![-,-]\!]$

Definition

$\widehat{\mathrm{d}} : \widehat{\mathrm{BR}}(\mathcal{A}) \to \oplus_{n\geq 0} \mathrm{Hom}_{\Bbbk}(\mathcal{A}^{\otimes(n+1)}, \mathcal{A}^{\otimes(n+1)})$ is defined in deg 0 by

$$\widehat{\mathrm{d}}(\bar{a}) : \mathcal{A} \to \mathcal{A}, \quad \widehat{\mathrm{d}}(\bar{a}) := -\mathrm{m} \circ [\![a,-]\!] , \qquad (a \in \mathcal{A} \text{ is a lift of } \bar{a} \in \mathcal{A}_\sharp)$$

and in degree $n \geq 1$ by ($\{\!\{-\}\!\} \in \widehat{\mathrm{BR}}(\mathcal{A})_n$)

$$\widehat{\mathrm{d}}(\{\!\{-\}\!\}) := \sum_{s\in\mathbb{Z}_{n+1}} (-1)^{ns}\sigma^s \circ (\{\!\{-\}\!\} \otimes \mathrm{Id}_{\mathcal{A}}) \circ (\mathrm{Id}_{\mathcal{A}}^{\otimes(n-1)} \otimes [\![-,-]\!]) \circ \sigma^{-s}$$
$$+ (-1)^n \sum_{s\in\mathbb{Z}_{n+1}} (-1)^{ns}\sigma^s \circ ([\![-,-]\!] \otimes \mathrm{Id}_{\mathcal{A}}^{\otimes(n-1)}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \{\!\{-\}\!\}) \circ \sigma^{-s} .$$

The cohomology

Recall that we work over $\mathbb{k}$, although everything works over $B \hookrightarrow \mathcal{A}$

Theorem

Consider the operation $\widehat{\mathrm{d}}$ from the previous definition

1. *For any $n \geq 0$, $\widehat{\mathrm{d}}(\widehat{\mathrm{BR}}(\mathcal{A})_n) \subset \widehat{\mathrm{BR}}(\mathcal{A})_{n+1}$.*
   Hence, we can view $\widehat{\mathrm{d}}$ as a degree $+1$ endomorphism of $\widehat{\mathrm{BR}}(\mathcal{A})$.
2. *The linear map $\widehat{\mathrm{d}}$ is a square-zero differential on $\widehat{\mathrm{BR}}(\mathcal{A})$.*

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Proof.

1. Direct computation. 2. The terms either cancel out or can be grouped in $\mathbb{D}\mathrm{Jac}$.
E.g. $\widehat{\mathrm{d}}^2(\bar{a})(b \otimes c) = -(\mathrm{m} \otimes \mathrm{Id})\mathbb{D}\mathrm{Jac}(a, b, c) + (\mathrm{Id} \otimes \mathrm{m})\mathbb{D}\mathrm{Jac}(b, a, c)$ □

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$\rightsquigarrow$ completed double Poisson cohomology: $\widehat{\mathrm{dPH}}(\mathcal{A}) = \mathrm{H}(\widehat{\mathrm{BR}}(\mathcal{A}), \widehat{\mathrm{d}})$

Also: analog for double Lie algebras, see §4.1.4

Relating double Poisson cohomologies

Theorem

If $[\![-,-]\!] = \mu_2(P)$ *is a double Poisson bracket for some* $P \in (T^*\mathcal{A})_2$*, then the maps* $(\mu_n)_{n\geq 0}$ *define a morphism of complexes* $(\mathbb{T}^*\mathcal{A})_\sharp \longrightarrow \widehat{\mathrm{BR}}(\mathcal{A})$ *endowed with the Pichereau-Van de Weyer differential* d_P *and the 'new'* $\widehat{\mathrm{d}}$
Thus, we obtain a $\Bbbk$*-linear map* $\mathrm{dPH}(\mathcal{A}) \to \widehat{\mathrm{dPH}}(\mathcal{A})$ *in cohomology.*

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{T}^*\mathcal{A})_{\sharp,0} \arrow[r, "\mathrm{d}_P"] \arrow[d, "\mu_0"'] & (\mathbb{T}^*\mathcal{A})_{\sharp,1} \arrow[r, "\mathrm{d}_P"] \arrow[d, "\mu_1"'] & (\mathbb{T}^*\mathcal{A})_{\sharp,2} \arrow[r, "\mathrm{d}_P"] \arrow[d, "\mu_2"'] & (\mathbb{T}^*\mathcal{A})_{\sharp,3} \arrow[r, "\mathrm{d}_P"] \arrow[d, "\mu_3"'] & \cdots \\
0 \arrow[r] & \widehat{\mathrm{BR}}(\mathcal{A})_0 \arrow[r, "\widehat{\mathrm{d}}"] & \widehat{\mathrm{BR}}(\mathcal{A})_1 \arrow[r, "-\widehat{\mathrm{d}}"] & \widehat{\mathrm{BR}}(\mathcal{A})_2 \arrow[r, "\widehat{\mathrm{d}}"] & \widehat{\mathrm{BR}}(\mathcal{A})_3 \arrow[r, "-\widehat{\mathrm{d}}"] & \cdots
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\end{tikzcd}
$$

Corollary

When all $(\mu_n)_{n\geq 0}$ are isomorphisms, $\mathrm{dPH}(\mathcal{A})$ and $\widehat{\mathrm{dPH}}(\mathcal{A})$ are isomorphic.

Chemla's formula

Proposition

Fix $[\![-,-]\!] \in \widehat{\mathrm{BR}}(\mathcal{A})_2$ *and* $\{\!\{-\}\!\} \in \widehat{\mathrm{BR}}(\mathcal{A})_n$, $n \geq 1$. *Then, the* $(n+1)$*-bracket* $\widehat{\mathrm{d}}(\{\!\{-\}\!\}) \in \widehat{\mathrm{BR}}(\mathcal{A})_{n+1}$ *satisfies*

$$\begin{aligned}
&\widehat{\mathrm{d}}(\{\!\{-\}\!\})(a_1 \otimes \ldots \otimes a_{n+1}) \\
&= \sum_{i=1}^{n+1} (-1)^{ni}\, \sigma^i \, \{\!\{a_{i+1}, \ldots, a_{n+1}, a_1, \ldots, a_{i-2}, [\![a_{i-1}, a_i]\!]\}\!\}_L \\
&\quad + \sum_{i=1}^{n+1} (-1)^{ni}\, \sigma^{i-1} \, [\![a_i, \{\!\{a_{i+1}, \ldots, a_{n+1}, a_1, \ldots, a_{i-1}\}\!\}]\!]_L
\end{aligned}$$

Proof.

Direct calculation. □

$\rightsquigarrow$ related to the construction of [Chemla,'20; 1712.05619] for the double Lie-Rinehart algebra cohomology on $\Omega^1_{\mathcal{A}}$ with

$\rho : \Omega^1_{\mathcal{A}} \to \mathbb{D}\mathrm{er}(\mathcal{A})$, $\quad \rho(\mathrm{d}\, a) := [\![a, -]\!]$, $\{\!\{\mathrm{d}\, a, \mathrm{d}\, b\}\!\}_{\Omega^1_{\mathcal{A}}} := (\mathrm{d} \otimes \mathrm{Id}_{\mathcal{A}} + \mathrm{Id}_{\mathcal{A}} \otimes \mathrm{d}\,)\, [\![a, b]\!]$

Operations on representation algebras (1)

We follow Chapter 7 of [F-Valeri,2509.21232]

$S_k^{(1)} = \{\tau \in S_k \mid \tau(1) = 1\}$; Recall $\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj} \; \{\!\{a,b\}\!\}''_{il}$ [VdB,'08]

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Theorem

Given $\{\!\{-\}\!\} \in \widehat{\mathrm{BR}}(\mathcal{A})_k$, $\exists!\ \mathrm{tr}(\{\!\{-\}\!\}) \in \mathfrak{X}^k(\mathcal{A}_N)^{\mathrm{GL}_N}$ *satisfying*

$$\mathrm{tr}(\{\!\{-\}\!\})(a^1_{u_1v_1}, \dots, a^k_{u_kv_k}) = \sum_{\tilde\sigma \in S_k^{(1)}} \mathrm{sgn}(\tilde\sigma) \left\{\!\!\left\{a^1, a^{\tilde\sigma(2)}, \dots, a^{\tilde\sigma(k)}\right\}\!\!\right\}_{\tilde\sigma(u,v)}$$

for any $a^j \in \mathcal{A}$, $1 \leq u_j, v_j \leq N$ *with* $1 \leq j \leq k$, *and the notation*

$$\tilde\sigma(u,v) := (u_{\tilde\sigma(k)}v_1, u_1v_{\tilde\sigma(2)}, \dots, u_{\tilde\sigma(k-1)}v_{\tilde\sigma(k)}).$$

This defines a linear map $\mathrm{tr} : \widehat{\mathrm{BR}}(\mathcal{A})_k \longrightarrow \mathfrak{X}^k(\mathcal{A}_N)^{\mathrm{GL}_N}$ *whose restriction to* $\mu_k((\mathbb{T}^*\mathcal{A})_k)$ *coincides with* $\mathrm{tr} : (\mathbb{T}^*\mathcal{A})_\sharp \to \mathfrak{X}^k(\mathcal{A}_N)^{\mathrm{GL}_N}$.

Operations on representation algebras (2)

Theorem

Assume that $\{\!\{-,-\}\!\} \in \widehat{\mathrm{BR}}(\mathcal{A})_2$ *is a double Poisson bracket, and let* $\{-,-\} = \mathrm{tr}(\{\!\{-,-\}\!\})$ *denote the associated Poisson bracket on* $\mathcal{A}_N$*. Then, there is a morphism of complexes*

$$\mathrm{tr} : (\widehat{\mathrm{BR}}(\mathcal{A}), \widehat{\mathrm{d}}) \longrightarrow (\mathfrak{X}(\mathcal{A}_N), (-1)^{\bullet}\, \delta_{\mathcal{A}_N,\{-,-\}}),$$

which descends to a linear map $\widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE}(\mathcal{A}_N)$ *in cohomology. Furthermore, by* GL_N*-invariance, this descends to a linear map* $\widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE}(\mathcal{A}_N^{\mathrm{GL}_N})$ *in cohomology.*

Proof.

Commutativity of

$$\begin{array}{ccc} \widehat{\mathrm{BR}}(\mathcal{A})_{k-1} & \xrightarrow{\widehat{\mathrm{d}}} & \widehat{\mathrm{BR}}(\mathcal{A})_k \\ \mathrm{tr}\downarrow & & \downarrow\mathrm{tr} \\ \mathfrak{X}^{k-1}(\mathcal{A}_N) & \xrightarrow{(-1)^{k-1}\,\delta^{k-1}} & \mathfrak{X}^k(\mathcal{A}_N) \end{array}$$

$\square$

Operations on representation algebras (3)

Corollary

Assume that $P \in (\mathbb{T}^*\mathcal{A})_{\sharp,2}$ satisfies $\{P,P\}_{\mathrm{SN}} = 0$. Then, the following diagrams are commutative:

$$\begin{tikzcd}
\mathrm{dPH}(\mathcal{A}) \arrow[r, "\mu"] \arrow[d, "\mathrm{tr}"'] & \widehat{\mathrm{dPH}}(\mathcal{A}) \arrow[d, "\mathrm{tr}"] \\
\mathrm{PH}(\mathcal{A}_N) \arrow[r, "\sim"] & \mathrm{H}_{CE}(\mathcal{A}_N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{dPH}(\mathcal{A}) \arrow[r, "\mu"] \arrow[d, "\mathrm{tr}"'] & \widehat{\mathrm{dPH}}(\mathcal{A}) \arrow[d, "\mathrm{tr}"] \\
\mathrm{PH}(\mathcal{A}_{\mathbf{n}}^{\mathrm{GL}_N}) \arrow[r, "\sim"] & \mathrm{H}_{CE}(\mathcal{A}_N^{\mathrm{GL}_N})
\end{tikzcd}$$

where we use the differentials d_P, $\widehat{\mathrm{d}}$, $\mathrm{d}_{\mathrm{tr}(P)} = [\mathrm{tr}(P), -]_{\mathrm{SN}}$ and δ_{CE} on the top left, top right, bottom left and bottom right, respectively.

!! The vertical trace maps are, in general, neither injective nor surjective

Example 1

We follow Chapter 6 of [F-Valeri,2509.21232]

On $\mathbb{k}[x]$, (for $\lambda, \mu, \nu \in \mathbb{k}$)

$$[\![x, x]\!] = \nu(x^2 \otimes x - x \otimes x^2) + \mu(x^2 \otimes 1 - 1 \otimes x^2) + \lambda(x \otimes 1 - 1 \otimes x)$$

It is Poisson iff $\lambda\nu - \mu^2 = 0$ [Powell,'16].

All (μ_n) are iso, and $[\![-, -]\!] = \mu_2(P)$ for

$$P = \lambda\, x\partial_x\partial_x + \mu\, x^2\partial_x\partial_x + \nu\, x^2\partial_x x\partial_x\,, \quad \partial_x \in \mathbb{D}\mathrm{er}(\mathbb{k}[x]),\ x \mapsto 1 \otimes 1$$

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We assume $\lambda\nu - \mu^2 = 0$ from now on.
$\llbracket -, - \rrbracket$ is also a double Poisson bracket on any $\Bbbk[x]/(x^r)$ (but no P!)

Results on $\mathbb{k}[x]$ and $\mathbb{k}[x]/(x^r)$

Proposition

- $\widehat{\mathrm{dPH}}^0(\mathbb{k}[x]) = \mathbb{k}[x]$.
- $\widehat{\mathrm{dPH}}^1(\mathbb{k}[x]) = \mathbb{k}\,(\nu\theta_2 + 2\mu\theta_1 + \lambda\theta_0)$, *where* $\theta_j \in \mathrm{Der}(\mathbb{k}[x])\ x \mapsto x^j$
  $\mathrm{dPH}^1(\mathbb{k}[x]) = \mathbb{k}\left(\nu x^2\partial_x + 2\mu x\partial_x + \lambda\partial_x\right)$
- $\widehat{\mathrm{dPH}}^2(\mathbb{k}[x]) = \mathrm{dPH}^2(\mathbb{k}[x]) = \{0\}$.

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$[\![-,-]\!]$ is not differential on $\mathbb{k}[x]/(x^r)$ $\rightsquigarrow$ only $\widehat{\mathrm{dPH}}$

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- $\widehat{\mathrm{dPH}}^0(\mathbb{k}[x]/(x^r)) = \mathbb{k}[x]/(x^r)$.
- $\widehat{\mathrm{dPH}}^1(\mathbb{k}[x]/(x^r)) = \mathbb{k}\,\theta_2$ *if* $\nu \neq 0$ *and* $\mu = \lambda = 0$.
  $\widehat{\mathrm{dPH}}^1(\mathbb{k}[x]/(x^r)) = \{0\}$ *in all other cases.* *(only possibility for* $r = 2$*)*
- $\widehat{\mathrm{dPH}}^2(\mathbb{k}[x]/(x^r)) = \mathbb{k}\,\{\!\{-,-\}\!\}_{2,0}$ *if* $\nu \neq 0$, $\mu = \lambda = 0$.
  $\widehat{\mathrm{dPH}}^2(\mathbb{k}[x]/(x^r)) = \{0\}$ *in all other cases.* *(only possibility for* $r = 2$*)*

Inducing the results

Recall $\Bbbk[x]_N \simeq \Bbbk[\mathfrak{gl}_N]$, thus $\mathrm{Spec}((\Bbbk[x]/(x^r))_N) \simeq \{Y \in \mathfrak{gl}_N \mid Y^r = 0_N\}$

E.g. for $\nu \neq 0$ and $\lambda = \mu = 0$ $(r > 2)$ we obtain:

- From $\widehat{\mathrm{dPH}}^0$: $\sum_{k\geq 0} \Bbbk\,\mathrm{tr}(x^k) \subset \mathrm{H}^0_{CE}(\mathcal{A}_N)$ (invariants/spectrum)

  $\rightsquigarrow \widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE}(\mathcal{A}_N)$ clearly not injective
- From $\widehat{\mathrm{dPH}}^1$: $\Bbbk\,\mathrm{tr}(\theta_2)$ where

$$\mathrm{tr}(\theta_2) : x_{ij} \mapsto \sum_{1\leq r\leq N} x_{ir}x_{rj}$$

- From $\widehat{\mathrm{dPH}}^2$ <u>in truncated case only</u>: $\Bbbk\,\mathrm{tr}(\{\!\{-,-\}\!\}_{2,0})$ where

$$\mathrm{tr}(\{\!\{-,-\}\!\}_{2,0}) : (x_{ij}, x_{pq}) \mapsto \sum_{1\leq r\leq N} (x_{pr}x_{rj}\delta_{iq} - \delta_{pj}x_{ir}x_{rq})$$

Nondegenerate constant case (1)

Let $\ell \geq 2$. Put $\mathcal{A} = \Bbbk\langle u_1, \dots, u_\ell \rangle$
Fix skewsymmetric $C = (C_{ij}) \in \mathrm{GL}_\ell(\Bbbk)$

Lemma

There is a unique double Poisson bracket $\{\!\{-,-\}\!\}$ on $\mathcal{A}$ satisfying

$$\{\!\{u_i, u_j\}\!\} = C_{ij}\, 1 \otimes 1\,, \qquad\qquad 1 \leq i, j \leq \ell$$

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Analogous to [De Sole-Kac-Valeri,'15; Thm. 2.18] :

Theorem

$$\dim\left(\widehat{\mathrm{dPH}}^n(\mathcal{A})\right) = \dim\left(\mathrm{dPH}^n(\mathcal{A})\right) = \delta_{n0}\,, \quad n \geq 0\,.$$

Remark: induced Poisson bracket $\mathrm{tr}(\{\!\{-,-\}\!\})$ is non-degenerate so $\mathrm{PH}^n(\mathcal{A}_N) = \delta_{n0}\Bbbk\ \forall N$ and $\mathrm{dPH}(\mathcal{A}) \to \mathrm{PH}(\mathcal{A}_N)$ is an isomorphism.

Nondegenerate constant case (2)

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$$\dim\left(\widehat{\mathrm{dPH}}^n(\mathcal{A})\right) = \dim\left(\mathrm{dPH}^n(\mathcal{A})\right) = \delta_{n0}\,, \quad n \geq 0\,.$$

Proof.

1. All μ_n are iso so $\widehat{\mathrm{dPH}}^n(\mathcal{A}) \simeq \mathrm{dPH}^n(\mathcal{A})$
2. If $\bar{f} \in \mathcal{A}_\sharp$ s.t. $\widehat{\mathrm{d}}(\bar{f}) = 0$, then $\bar{f} \in \mathbb{k}$ (this gives $\widehat{\mathrm{dPH}}^0$)

$\rightsquigarrow$ this uses Euler vector field $\mathrm{E} : f \mapsto \deg(f)\, f$

$\square$

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3. Extend E to $L_\mathrm{E} : \widehat{\mathrm{BR}}(\mathcal{A}) \to \widehat{\mathrm{BR}}(\mathcal{A})$ invertible outside $\Bbbk \subset \widehat{\mathrm{BR}}(\mathcal{A})_0$

Define contraction operator $\iota_C : \widehat{\mathrm{BR}}(\mathcal{A})_\bullet \to \widehat{\mathrm{BR}}(\mathcal{A})_{\bullet-1}$ using C^{-1}

Get *homotopy operator* $h_{\mathrm{E},C} = (L_\mathrm{E})^{-1} \circ \iota_C : \bigoplus_{n>0} \widehat{\mathrm{BR}}(\mathcal{A})_n \to \widehat{\mathrm{BR}}(\mathcal{A})$

□

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If $\{\!\{-\}\!\} \in \widehat{\mathrm{BR}}(\mathcal{A})_n$, $n > 0$, $(\widehat{\mathrm{d}} \circ h_{\mathrm{E},C} - h_{\mathrm{E},C} \circ \widehat{\mathrm{d}})(\{\!\{-\}\!\}) = (-1)^{n+1}\{\!\{-\}\!\}$

4. If $\widehat{\mathrm{d}}(\{\!\{-\}\!\}) = 0$, it must be a coboundary! □

Plan for the talk

1. Motivation and classical constructions
2. NC Poisson geometry and a first cohomology theory
3. Completed double Poisson algebra cohomology
4. **Final remarks**

Double quasi-Poisson cohomology (1)

This follows Sections 5.2 and 7.3.2 of [F-Valeri,2509.21232]

Definition

$[\![-,-]\!]$ is a double quasi-Poisson bracket if the associated triple bracket $\mathbb{D}\mathrm{Jac}$ satisfies for a fixed $q \in \Bbbk^\times$ and any $a, b, c \in \mathcal{A}$

$$\mathbb{D}\mathrm{Jac}(a,b,c) = q\Big(ca \otimes b \otimes 1 - ca \otimes 1 \otimes b - c \otimes ab \otimes 1 + c \otimes a \otimes b \\ - a \otimes b \otimes c + a \otimes 1 \otimes bc + 1 \otimes ab \otimes c - 1 \otimes a \otimes bc \Big).$$

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Proposition

If $[\![-,-]\!]$ is a double quasi-Poisson bracket, then, the (degree $+1$) operations $\widehat{\mathrm{d}} : \widehat{\mathrm{BR}}(\mathcal{A}) \to \widehat{\mathrm{BR}}(\mathcal{A})$ given as in the Poisson case link *define a square-zero differential on the complex $\widehat{\mathrm{BR}}(\mathcal{A})$.*

Proof.

One has to carefully replace all $\mathbb{D}\mathrm{Jac} = 0$ from the Poisson case by the above equality and cancel out terms. $\square$

Double quasi-Poisson cohomology (2)

Assume that $\llbracket -,- \rrbracket \in \widehat{\mathrm{BR}}(\mathcal{A})_2$ is a double quasi-Poisson bracket.

Recall $\mathrm{tr}: \widehat{\mathrm{BR}}(\mathcal{A})_k \longrightarrow \mathfrak{X}^k(\mathcal{A}_N)^{\mathrm{GL}_N}$, $\forall\, k$

Proposition

The skewsymmetric biderivation $\mathrm{tr}(\llbracket -,- \rrbracket)$ defined on $\mathcal{A}_N$ is a quasi-Poisson bracket in the sense of [Alekseev–Kosmann-Schwarzbach–Meinrenken,'02]

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Following [AKSM,'02], we get quasi-Poisson cohomology by restricting to GL_N-invariant multivector fields: $\mathrm{H}_{CE;\mathrm{GL}_N}(\mathcal{A}_N) := \mathrm{H}(\mathfrak{X}(\mathcal{A}_N)^{\mathrm{GL}_N}, \delta)$

Theorem

The differential $\widehat{\mathrm{d}}$ *on* $\widehat{\mathrm{BR}}(\mathcal{A})$ *(defined using* $\llbracket -,- \rrbracket$*) and the CE differential* δ *on* $\mathfrak{X}^k(\mathcal{A}_N)^{\mathrm{GL}_N}$ *(defined using* $\mathrm{tr}(\llbracket -,- \rrbracket)$*) induce*

$$\mathrm{tr} : (\widehat{\mathrm{BR}}_B(\mathcal{A}), \widehat{\mathrm{d}}) \longrightarrow (\mathfrak{X}(\mathcal{A}_N)^{\mathrm{GL}_N}, (-1)^{\bullet}\,\delta),$$

which descends to a linear map $\widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE;\mathrm{GL}_N}(\mathcal{A}_N)$ *in cohomology. Furthermore, this descends to a linear map* $\widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE}(\mathcal{A}_N^{\mathrm{GL}_N})$.

The gauge element

We follow Sections 5.1, 5.3 and 7.3 of [F-Valeri,2509.21232]

Construction from [Alekseev-Kawazumi-Kuno-Naef,'20] in the simplest case

Lemma (Van den Bergh,'08)

The gauge element $\Delta \in \mathbb{D}\mathit{er}(\mathcal{A})$, $a \mapsto a \otimes 1 - 1 \otimes a$ *induces infinitesimal vector fields of* $\mathrm{GL}_N \curvearrowright \mathcal{A}_N$ *under* tr.

Proof.

$\Delta_{ij}(a_{kl}) = a_{kj}\delta_{il} - \delta_{kj}a_{il} = [\mathtt{X}(a), E_{ji}]_{kl}$ $\square$

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Proof.

$\Delta_{ij}(a_{kl}) = a_{kj}\delta_{il} - \delta_{kj}a_{il} = [\mathtt{X}(a), E_{ji}]_{kl}$ □

Corollary

Any $Q = Q_1 \cdots Q_{k-1}\delta \in (\mathbb{T}^*\mathcal{A})_k$ *with* $Q_j \in \mathbb{D}\mathrm{er}(\mathcal{A})$ *is such that* $\mathrm{tr}(Q) \in \mathfrak{X}^k(\mathcal{A}_N)$ *vanishes on* $\mathcal{A}_N^{\mathrm{GL}_N}$.

Proof.

Invariant elements are sent to zero under the infinitesimal action. □

Gauged double Poisson (1)

We introduce $(k \geq 1)$ $\iota_k^\Delta : (\mathbb{T}^*\mathcal{A})_{k-1} \to (\mathbb{T}^*\mathcal{A})_{\sharp,k}, \quad \alpha \mapsto (\alpha\Delta)_\sharp$

- By the corollary, $\mathrm{tr} \circ \iota_k^\Delta(\alpha)$ is the zero multivector on $\mathcal{A}_N^{\mathrm{GL}_N}$

Let $\mathcal{D}_\mathcal{A}^k := \operatorname{coker} \iota_k^\Delta$ with projection $\pi_k^\Delta : (\mathbb{T}^*\mathcal{A})_{\sharp,k} \to \mathcal{D}_\mathcal{A}^k$.
For $k = 0$, $\iota_0^\Delta : 0 \to \mathcal{A}_\sharp$ and $\pi_0^\Delta = \mathrm{Id}_{\mathcal{A}_\sharp}$.

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For $k = 0$, $\iota_0^\Delta : 0 \to \mathcal{A}_\sharp$ and $\pi_0^\Delta = \mathrm{Id}_{\mathcal{A}_\sharp}$.

- If $P \in (\mathbb{T}^*\mathcal{A})_{\sharp,2}$ is Poisson, d_P restricts to $\mathcal{D}_\mathcal{A}$

$\rightsquigarrow$ gauged double Poisson cohomology $\mathrm{gdPH}(\mathcal{A}) = \mathrm{H}(\mathcal{D}_\mathcal{A}, \mathrm{d}_P)$

Gauged double Poisson (2)

If $[\![-,-]\!] = \mu_2(P)$ is a differential double Poisson bracket:
$\widehat{\mathrm{d}}$ restricts to $\widehat{\mathcal{D}}^k_{\mathcal{A}} := \operatorname{coker} \mu_k \circ \iota^\Delta_k$

$$\begin{tikzcd}
 & & & 0 \arrow[dl, "\iota_0^\Delta"'] & & (\mathbb{T}^*\mathcal{A})_0 \arrow[dl, "\iota_1^\Delta"'] & & (\mathbb{T}^*\mathcal{A})_1 \arrow[dl, "\iota_2^\Delta"'] & & (\mathbb{T}^*\mathcal{A})_2 \arrow[dl, "\iota_3^\Delta"'] \\
 & 0 \arrow[r] & (\mathbb{T}^*\mathcal{A})_{\sharp,0} \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathbb{T}^*\mathcal{A})_{\sharp,1} \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathbb{T}^*\mathcal{A})_{\sharp,2} \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathbb{T}^*\mathcal{A})_{\sharp,3} \arrow[r] \arrow[dl] \arrow[dd] & \cdots \\
0 \arrow[r] & \mathcal{D}^0_{\mathcal{A}} \arrow[rr] \arrow[dd] & & \mathcal{D}^1_{\mathcal{A}} \arrow[rr] \arrow[dd] & & \mathcal{D}^2_{\mathcal{A}} \arrow[rr] \arrow[dd] & & \mathcal{D}^3_{\mathcal{A}} \arrow[rr] \arrow[dd] & & \cdots \\
 & 0 \arrow[r] & \widehat{\mathrm{BR}}(\mathcal{A})_0 \arrow[rr] \arrow[dl] & & \widehat{\mathrm{BR}}(\mathcal{A})_1 \arrow[rr] \arrow[dl] & & \widehat{\mathrm{BR}}(\mathcal{A})_2 \arrow[rr] \arrow[dl] & & \widehat{\mathrm{BR}}(\mathcal{A})_3 \arrow[r] \arrow[dl] & \cdots \\
0 \arrow[r] & \widehat{\mathcal{D}}^0_{\mathcal{A}} \arrow[rr] & & \widehat{\mathcal{D}}^1_{\mathcal{A}} \arrow[rr] & & \widehat{\mathcal{D}}^2_{\mathcal{A}} \arrow[rr] & & \widehat{\mathcal{D}}^3_{\mathcal{A}} \arrow[rr] & & \cdots
\end{tikzcd}$$

$\rightsquigarrow$ completed gauged double Poisson cohomology $\widehat{\mathrm{gdPH}}(\mathcal{A}) = \mathrm{H}(\widehat{\mathcal{D}}_{\mathcal{A}}, \widehat{\mathrm{d}})$

Gauged double Poisson (3)

If $\llbracket -,- \rrbracket = \mu_2(P)$ is a differential double Poisson bracket:

Theorem

The following diagram is commutative:

$$\begin{tikzcd}
& \mathrm{dPH}(\mathcal{A}) \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & \widehat{\mathrm{dPH}}(\mathcal{A}) \arrow[dl] \arrow[dd] \\
\mathrm{gdPH}(\mathcal{A}) \arrow[rr] \arrow[dd] & & \widehat{\mathrm{gdPH}}(\mathcal{A}) \arrow[dd] & \\
& \mathrm{PH}(\mathcal{A}_N) \arrow[rr, dashed] \arrow[dl, dashed] & & \mathrm{H}_{CE}(\mathcal{A}_N) \arrow[dl] \\
\mathrm{PH}(\mathcal{A}_N^{\mathrm{GL}_N}) \arrow[rr] & & \mathrm{H}_{CE}(\mathcal{A}_N^{\mathrm{GL}_N}) &
\end{tikzcd}$$

Gauged double Poisson (4)

Example

For any one of the (nonzero) double Poisson bracket defined on $\Bbbk[x]$, each map $\mathrm{d}_P : \mathcal{D}^k_{\Bbbk[x]} \to \mathcal{D}^{k+1}_{\Bbbk[x]}$ is the zero map.

One can deduce that $\mathrm{gdPH}^k(\Bbbk[x]) = \mathcal{D}^k_{\Bbbk[x]}$ for all $k \geq 0$, and

1. $\mathrm{gdPH}^0(\Bbbk[x]) = (\mathbb{T}^*\Bbbk[x])_{0,\sharp} = \Bbbk[x]$;
2. $\mathrm{gdPH}^1(\Bbbk[x]) = (\mathbb{T}^*\Bbbk[x])_{1,\sharp} = \oplus_{j\geq 0} \Bbbk(x^j \partial_x)_\sharp$;
3. $\mathrm{gdPH}^{2\ell}(\Bbbk[x]) = 0$ and $\mathrm{gdPH}^{2\ell+1}(\Bbbk[x]) = \oplus_{j\geq 0}\Bbbk(x^j \partial_x^{2\ell+1})_\Delta$ for $\ell \geq 1$.

Gauged double Poisson (5)

Definition

$P \in (\mathbb{T}^*\mathcal{A})_{\sharp,2}$ is gauged Poisson if there exists $\alpha \in (\mathbb{T}^*\mathcal{A})_2$ such that $\{P, P\}_{\mathrm{SN}} = (\Delta\alpha)_\sharp \in (\mathbb{T}^*\mathcal{A})_{\sharp,3}$. The corresponding double bracket $[\![-,-]\!] = \mu_2(P)$ is then said to be a double gauged Poisson bracket.

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Proposition

Assume that $P \in (\mathbb{T}^*\mathcal{A})_{\sharp,2}$ is a gauged Poisson element. Then the linear operation (of degree $+1$) d_P on $(\mathbb{T}^*\mathcal{A})_\sharp$ descends to a square-zero differential on $\mathcal{D}_\mathcal{A}$.

Proof.

One computes $\mathrm{Im}(\mathrm{d}_P^2) \subseteq \mathrm{Im}(\iota^\Delta)$. Then, recall $\mathcal{D}^k_\mathcal{A} := \operatorname{coker} \iota^\Delta_k$ $\square$

Gauged double Poisson (6)

Proposition

If $P \in (\mathbb{T}^*\mathcal{A})_{\sharp,2}$ *is gauged Poisson, then* $\mathrm{tr}(P) \in \mathfrak{X}^2(\mathcal{A}_N)^{\mathrm{GL}_N}$ *induces a Poisson bivector on* $\mathcal{A}_N^{\mathrm{GL}_N}$.

Proof.

$\{P,P\}_{\mathrm{SN}} = (\Delta\alpha)_\sharp$ implies $[\mathrm{tr}(P),\mathrm{tr}(P)]_{\mathrm{SN}} = \sum_{i,j}(E_{ji})_{\mathcal{A}_N} \wedge \alpha_{ji}$. □

Theorem

There is a morphism of complexes

$$\mathrm{tr} : (\mathcal{D}_{\mathcal{A}}, \mathrm{d}_P) \longrightarrow (\mathfrak{X}(\mathcal{A}_N^{\mathrm{GL}_N}), \mathrm{d}_{\mathrm{tr}(P)}),$$

which descends to a linear map $\mathrm{gdPH}(\mathcal{A}) \to \mathrm{PH}(\mathcal{A}_N^{\mathrm{GL}_N})$ *in cohomology.*

To go further: dPVA cohomology

In part 2 of arXiv:2509.21232 with Daniele Valeri (La Sapienza, IT), we give cohomological constructions for **double Poisson vertex algebras**

In part 3, we show that the dPA and dPVA cohomologies are related under the jet and quotient functors (see [Bozec-F-Moreau,'25]).

It is time to make calculations for those theories...

Thank you for listening !

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