Double Poisson vertex algebra cohomology

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Plan for the talk

1. **Motivation**
2. PVA cohomologies
3. dPA and dPVA
4. dPVA cohomology

Based on arXiv:2509.21232 with Daniele Valeri (La Sapienza, IT)

Classical Poisson cohomology

Fix a *commutative associative* $\mathbb{k}$-algebra A $\qquad (\otimes := \otimes_{\mathbb{k}})$

Recall $(A, \{-,-\})$ is a Poisson algebra if $\{-,-\} : A \otimes A \to A$ linear s.t.

(skewsymmetry) $\{x, y\} = -\{y, x\}$

(Leibniz rules) $\{x, yz\} = y\{x, z\} + \{x, y\}z\,, \quad \{xy, z\} = x\{y, z\} + \{x, z\}y$

(Jacobi identity) $\{x, \{y, z\}\} - \{y, \{x, z\}\} = \{\{x, y\}, z\}$

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$\mathfrak{X}^n(A) \subset \mathrm{Hom}(\wedge^n A, A)$ spanned by elements P s.t.
$P(ab, a_2, \dots, a_n) = aP(b, a_2, \dots, a_n) + bP(a, a_2, \dots, a_n)\,.$

$$\begin{aligned}\delta^n_{CE}(P)(x_1,\dots,x_{n+1}) &= \sum_{1\le i\le n+1} (-1)^{i+1}\{x_i, P(x_1, \overset{i}{\check{\dots}}, x_{n+1})\}\\ &+ \sum_{1\le i<j\le n+1} (-1)^{i+j} P(\{x_i,x_j\}, x_1, \overset{i}{\check{\dots}}, \overset{j}{\check{\dots}}, x_{n+1})\,.\end{aligned}$$

From CE differential $\rightsquigarrow$ $\mathrm{H}_{CE}(A) := \mathrm{H}(\mathfrak{X}(A), \delta_{CE})$.

Kontsevich-Rosenberg principle

Field $\Bbbk$ char. 0, $\overline{\Bbbk} = \Bbbk$

Following [Kontsevich, '93] and [Kontsevich-Rosenberg, '99] $(N \in \mathbb{N}^{\times})$

$$\begin{array}{ccc} \text{associative } \Bbbk\text{-algebra} & \to & \text{commutative } \Bbbk\text{-algebra} \\ \mathcal{A} & \longrightarrow & \mathcal{A}_N := \Bbbk[\mathrm{Rep}(\mathcal{A}, N)] \end{array}$$

$\mathcal{A}_N$ is generated by symbols a_{ij}, $\forall a \in \mathcal{A}$, $1 \leq i, j \leq N$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Goal: Find a property P_{nc} on $\mathcal{A}$ that gives the usual property P on $\mathcal{A}_N$ for all $N \in \mathbb{N}^{\times}$

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NC Poisson cohomology $\Rightarrow$ Poisson cohomology [Talk, NA Day Online, 22/12/25]

Today: NC PVA cohomology $\Rightarrow$ PVA cohomology

Plan for the talk

1. Motivation
2. **PVA cohomologies**
3. dPA and dPVA
4. dPVA cohomology

PVA

Fix a *differential commutative associative* $\Bbbk$-algebra V, $\partial \in \mathrm{Der}(V)$

A *λ-bracket* on V is a linear map $\{-_\lambda-\}: V \otimes V \to V[\lambda]$ satisfying

(skewsymmetry)	$\{a_\lambda b\} = -(\|_{x=\partial}\{b_{-\lambda-x}a\})$
(Leibniz rules)	$\{a_\lambda bc\} = b\{a_\lambda c\} + \{a_\lambda b\}c\,,$
	$\{ab_\lambda c\} = \{a_{\lambda+x}c\}(\|_{x=\partial}b) + \{b_{\lambda+x}c\}(\|_{x=\partial}a)$
(sesquilinearity)	$\{\partial a_\lambda b\} = -\lambda\{a_\lambda b\}\,, \qquad \{a_\lambda \partial b\} = (\lambda+\partial)\{a_\lambda b\}$

$(V, \{-_\lambda-\})$ is a *Poisson vertex algebra (PVA)* when also

(Jacobi identity) $\qquad \{a_\lambda\{b_\mu c\}\} - \{b_\mu\{a_\lambda c\}\} - \{\{a_\lambda b\}_{\lambda+\mu}c\} = 0$

PVA

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Example

If $(A, \{-,-\})$ is a Poisson algebra, then its jet algebra $J_\infty A$ is a PVA for the λ-bracket uniquely determined by: $\forall a, b \in A$, $\{a_\lambda b\} = \{a, b\}$.

Big picture about PVA cohomologies

Mainly from [De Sole-Kac, 2013]

- $\widetilde{\Gamma}(V)$ space of *basic cochains*
  $\rightsquigarrow$ basic PVA cohomology $\mathrm{H}_{\mathrm{bas}}(V)$
- $\Gamma(V) = \widetilde{\Gamma}(V)/\partial\widetilde{\Gamma}(V)$
  $\rightsquigarrow$ reduced PVA cohomology $\mathrm{H}_{\mathrm{red}}(V)$
- $C^n(V)$ space of n-λ-*brackets*
  $\rightsquigarrow$ variational PVA cohomology $\mathrm{P}_{\mathrm{v}}\mathrm{H}(V)$

$$\begin{tikzcd} \mathrm{H}_{\mathrm{bas}}(V) \arrow[r] & \mathrm{H}_{\mathrm{red}}(V) \arrow[d] \\ & \mathrm{P}_{\mathrm{v}}\mathrm{H}(V) \end{tikzcd}$$

Vertical map obtained from an injection $\Gamma(V) \hookrightarrow C(V)$
For an algebra of differential polynomials: iso thus $\mathrm{H}_{\mathrm{red}}(V) \simeq \mathrm{P}_{\mathrm{v}}\mathrm{H}(V)$

Basic cochains $\widetilde{\Gamma}(V)$

degree 0: $\widetilde{\Gamma}^0(V) = V$

degree $n \geq 1$: $X \in \widetilde{\Gamma}^n(V)$ is a basic n-cochain when

$$X : V^{\otimes n} \to V[\lambda_1, \ldots, \lambda_n]\,, \quad a_1 \otimes \cdots \otimes a_n \mapsto X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_n)\,,$$

satisfies skewsymmetry + Leibniz rules + sesquilinearity as

$$X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_n) = -\operatorname{sgn}(\tau) X_{\lambda_{\tau(1)},\ldots,\lambda_{\tau(n)}}(a_{\tau(1)}, \ldots, a_{\tau(n)}),\ \ \forall \tau \in S_n;$$

$$X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_{i-1}, bc, a_{i+1}, \ldots, a_n)$$
$$= X_{\lambda_1,\ldots,\lambda_{i-1},\lambda_i+x,\lambda_{i+1},\ldots\lambda_n}(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)(|_{x=\partial} c)$$
$$+ X_{\lambda_1,\ldots,\lambda_{i-1},\lambda_i+x,\lambda_{i+1},\ldots\lambda_n}(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n)(|_{x=\partial} b);$$

$$X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_{i-1}, \partial a_i, a_{i+1}, \ldots, a_n)$$
$$= -\lambda_i X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n).$$

Basic cochains $\widetilde{\Gamma}(V)$

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$$= -\lambda_i X_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n).$$

- $\exists$ concatenation product $\widetilde{\Gamma}^n(V) \otimes \widetilde{\Gamma}^\ell(V) \to \widetilde{\Gamma}^{n+\ell}(V)$
- $\partial \curvearrowright \widetilde{\Gamma}^n(V)$ by $(\partial X)_{\underline{\lambda}}(\underline{a}) := (\lambda_1 + \cdots + \lambda_m + \partial) X_{\underline{\lambda}}(\underline{a})$

The differential on $\widetilde{\Gamma}(V)$

Assume that $(V, \{-_\lambda-\})$ is a PVA.

Definition

For every $X \in \widetilde{\Gamma}^n(V)$ let $\tilde{\delta}(X) : V^{\otimes(n+1)} \to V[\lambda_1, \dots, \lambda_{n+1}]$ be

$$\begin{aligned}
&\tilde{\delta}(X)_{\lambda_1,\dots,\lambda_{n+1}}(a_1, \dots, a_{n+1}) \\
&= \sum_{i=1}^{n+1} (-1)^{i+1} \Big\{ a_i{}_{\lambda_i} X_{\lambda_1, \overset{i}{\check{\dots}}, \lambda_{n+1}} (a_1, \overset{i}{\check{\dots}}, a_{n+1}) \Big\} \\
&\quad + \sum_{1 \le i < j \le n+1} (-1)^{i+j} X_{\lambda_i+\lambda_j, \lambda_1, \overset{i}{\check{\dots}} \overset{j}{\check{\dots}}, \lambda_{n+1}} (\{a_i{}_{\lambda_i} a_j\}, a_1, \overset{i}{\check{\dots}} \overset{j}{\check{\dots}}, a_{n+1}) \,.
\end{aligned}$$

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Example

$n = 0, 1$: $\tilde{\delta}(c) = \{-_\lambda c\}$; $\tilde{\delta}(X)_{\lambda,\mu}(a, b) = \{a_\lambda X_\mu(b)\} - \{b_\mu X_\lambda(a)\} - X_{\lambda+\mu}(\{a_\lambda b\})$

Then $\tilde{\delta}^2(c)_{\lambda,\mu}(a, b) = \{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} - \{\{a_\lambda b\}_{\lambda+\mu} c\}$

The basic PVA cohomology

Theorem ([De Sole-Kac, 2013])

Let V be a PVA and let $\widetilde{\Gamma}(V)$ be the space of basic cochains.

(a) *The linear map $\tilde{\delta}$ gives a well-defined map $\tilde{\delta} : \widetilde{\Gamma}^n(V) \to \widetilde{\Gamma}^{n+1}(V)$.*

(b) *It commutes with the action of ∂ on $\widetilde{\Gamma}(V)$. Moreover, $\tilde{\delta}^2 = 0$.*

basic PVA cohomology

$$\mathrm{H}_{\mathrm{bas}}(V) = \mathrm{H}(\widetilde{\Gamma}(V), \tilde{\delta}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{H}^n_{\mathrm{bas}}(V)$$

$$\mathrm{H}^n_{\mathrm{bas}}(V) = \ker(\tilde{\delta}|_{\widetilde{\Gamma}^n(V)}) / \tilde{\delta}(\widetilde{\Gamma}^{n-1}(V))$$

The reduced PVA cohomology

By the theorem, $(\partial\widetilde{\Gamma}(V), \tilde{\delta}) \subset (\widetilde{\Gamma}(V), \tilde{\delta})$ is a subcomplex.

Then, form the complex $(\Gamma(V), \delta)$ where

$$\Gamma(V) = \widetilde{\Gamma}(V)/\partial\widetilde{\Gamma}(V) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \Gamma^n(V)\,, \quad \Gamma^n(V) = \widetilde{\Gamma}^n(V)/\partial\widetilde{\Gamma}^n(V)\,,$$

and $\delta : \Gamma(V) \to \Gamma(V)$ is induced by $\tilde{\delta}$.

Reduced PVA cohomology

$$\mathrm{H}_{\mathrm{red}}(V) = \mathrm{H}(\Gamma(V), \delta) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathrm{H}^n_{\mathrm{red}}(V)$$

poly-λ-brackets $C(V)$

degree 0: $C^0(V) = V_\sharp := V/\partial V$

degree $n \geq 1$: A n-λ-bracket is

$$\{-_{\lambda_1} - \cdots - _{\lambda_{n-1}} -\} : V^{\otimes n} \to V[\lambda_1, \dots, \lambda_{n-1}]$$

that satisfies skewsymmetry + Leibniz rules + sesquilinearity as

$$\{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}} a_n\} = \operatorname{sgn}(\tau)|_{\lambda_n = \lambda_n^\dagger} \{a_{\tau(1)\lambda_{\tau(1)}} \cdots a_{\tau(n-1)\lambda_{\tau(n-1)}} a_{\tau(n)}\}$$

$\forall \tau \in S_n$ where $\lambda_n^\dagger := -\lambda_1 - \cdots - \lambda_{n-1} - \partial$;

$$\begin{aligned}
&\{a_{1\lambda_1} \cdots bc_{\lambda_i} \dots a_{n-1\lambda_{n-1}} a_n\} \\
&\overset{1 \leqslant i \leqslant n-1}{=} \{a_{1\lambda_1} \cdots b_{\lambda_i + x} \dots a_{n-1\lambda_{n-1}} a_n\}(|_{x=\partial} c) \\
&\quad + \{a_{1\lambda_1} \cdots c_{\lambda_i + x} \dots a_{n-1\lambda_{n-1}} a_n\}(|_{x=\partial} b);
\end{aligned}$$

$$\{a_{1\lambda_1} \cdots {}_{\lambda_{i-1}} (\partial a_i)_{\lambda_i} \cdots a_{n-1\lambda_{n-1}} a_n\} \overset{1 \leqslant i \leqslant n-1}{=} -\lambda_i \{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}} a_n\}$$

$$\{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}} (\partial a_n)\} = (\lambda_1 \cdots + \lambda_{n-1} + \partial)\{a_{1\lambda_1} \cdots a_{n-1\lambda_{n-1}} a_n\}.$$

The differential on $C(V)$

Assume that $(V, [-_\lambda -])$ is a PVA.

Definition

For every $c := \{-_{\lambda_1} - \cdots - _{\lambda_{n-1}} -\} \in C^n(V)$ let $\mathrm{d}(c) : V^{\otimes n+1} \to V[\lambda_1, \dots, \lambda_n]$ be

$$\mathrm{d}(c)_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_{n+1}) = \sum_{s=1}^{n+1} (-1)^{s+1}|_{\lambda_{n+1}=\lambda_{n+1}^\dagger} [a_s{}_{\lambda_s} \{a_1{}_{\lambda_1} \overset{s}{\check{\cdots}} a_n{}_{\lambda_n} a_{n+1}\}]$$

$$+ \sum_{1 \le s < t \le n+1} (-1)^{s+t}|_{\lambda_{n+1}=\lambda_{n+1}^\dagger} \{[a_s{}_{\lambda_s} a_t]_{\lambda_s+\lambda_t} a_1{}_{\lambda_1} \overset{s}{\check{\cdots}} \overset{t}{\check{\cdots}} a_n{}_{\lambda_n} a_{n+1}\} .$$

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Example

$n = 0, 1$: $\mathrm{d}(\int a) = -[a_\lambda -]|_{\lambda=0} = -[\int a, -]$;
$\mathrm{d}(D)_\lambda(b, c) = [D(b)_\lambda c] + [b_\lambda D(c)] - D([b_\lambda c])$

Then $\mathrm{d}^2(\int a)_\lambda(b, c) = ([a_\mu[b_\lambda c]] - [b_\lambda[a_\mu c]] - [[a_\mu b]_{\lambda+\mu} c])\,|_{\mu=0}$

The variational PVA cohomology

Theorem ([De Sole-Kac, 2009&2013])

Let V be a PVA and let $C(V)$ be the space of poly-λ-brackets. Then, the linear map d gives a well-defined map $\mathrm{d}: C^n(V) \to C^{n+1}(V)$. Moreover, $\mathrm{d}^2 = 0$.

variational PVA cohomology

$$\mathrm{P_vH}(V) = H(C(V), \mathrm{d}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{P_vH}^n(V)$$

$$\mathrm{P_vH}^n(V) = \ker(\mathrm{d}|_{C^n(V)}) / \mathrm{d}(C^{n-1}(V))$$

$\mathrm{P_vH}^0(V) = \{\int a \in V_\sharp \mid [\int a, b] = 0 \, \forall b \in V\} =: \mathrm{Cas}(V)$ (centre of $V_\sharp \curvearrowright V$)
$\mathrm{P_vH}^1(V) = \mathrm{Der}((V, [-_\lambda -])) / \mathrm{Inn}(V)$

Plan for the talk

1. Motivation
2. PVA cohomologies
3. **dPA and dPVA**
   a) *dPA and their cohomology*
   b) *dPVA as NC versions of PVA*
4. dPVA cohomology

Double (Poisson) brackets

$\mathcal{A}$ is a unital associative fin. gen. algebra over $\Bbbk$, $\otimes := \otimes_{\Bbbk}$

Definition ([Van den Bergh,'08])

A double bracket is a $\Bbbk$-linear map $\{\!\{-,-\}\!\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t.

1. $\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^{\sigma}$ (cyclic skewsymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1)\{\!\{a,c\}\!\} + \{\!\{a,b\}\!\}(1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a)\{\!\{d,b\}\!\} + \{\!\{a,b\}\!\}(d \otimes 1)$ (inner derivation)

A double bracket is Poisson if "double Jacobiator" $\mathbb{D}\mathrm{Jac} : \mathcal{A}^{\otimes 3} \to \mathcal{A}^{\otimes 3}$,

$$\mathbb{D}\mathrm{Jac}(a,b,c) = \{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{b,\{\!\{c,a\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{c,\{\!\{a,b\}\!\}\}\!\}_L$$

is identically zero.

In the def, we use:
$(a_1 \otimes \cdots \otimes a_k)^{\sigma} = a_k \otimes a_1 \otimes \cdots \otimes a_{k-1}$ $(k=2)$ and $\{\!\{a, X' \otimes X''\}\!\}_L := \{\!\{a, X'\}\!\} \otimes X''$

We work over $\Bbbk$, although everything works over $B \hookrightarrow \mathcal{A}$

Relation to the Kontsevich-Rosenberg principle

$\mathcal{A}_N$ generated by symbols a_{ij}, $\forall a \in \mathcal{A}$, $1 \leq i, j \leq N$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Theorem ([Van den Bergh,'08])

If $\mathcal{A}$ has a double bracket $\{\!\{-,-\}\!\}$, then for any $N \geq 1$ $\mathcal{A}_N := \Bbbk[\mathrm{Rep}(\mathcal{A}, N)]$ has an antisymmetric biderivation $\{-,-\}$ defined by

$$\{a_{ij}, b_{kl}\} = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}\ .$$

If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}$ is a Poisson bracket.

Proof.

$\mathrm{Jac}(a_{ij}, b_{kl}, c_{uv}) = \mathbb{D}\mathrm{Jac}(a,b,c)_{uj,il,kv} - \mathbb{D}\mathrm{Jac}(a,c,b)_{kj,iv,ul}$ $\square$

multibrackets

A n-bracket on $\mathcal{A}$ is a linear map $\{\!\{-,\dots,-\}\!\}:\mathcal{A}^{\otimes n}\to\mathcal{A}^{\otimes n}$ such that (cyclic skewsymmetry) + (Leibniz rule), i.e.

$$\{\!\{a_1,a_2,\dots,a_n\}\!\}=(-1)^{n-1}\,\{\!\{a_2,\dots,a_n,a_1\}\!\}^{\sigma},$$
$$\{\!\{a_1,\dots,a_{n-1},bc\}\!\}=b\,\{\!\{a_1,\dots,a_{n-1},c\}\!\}+\{\!\{a_1,\dots,a_{n-1},b\}\!\}\,c\,.$$

$\rightsquigarrow$ Leibniz rule in each entry

$\mathrm{BR}(\mathcal{A})_n=\mathrm{span}_{\Bbbk}\{$all n-brackets$\}$, $\mathrm{BR}(\mathcal{A}):=\oplus_{n\geq 1}\mathrm{BR}(\mathcal{A})_n$

Completed in degree 0 : $\widehat{\mathrm{BR}}(\mathcal{A})=\oplus_{n\geq 0}\widehat{\mathrm{BR}}(\mathcal{A})_n$ with $\widehat{\mathrm{BR}}(\mathcal{A})_0=\mathcal{A}_\sharp:=\mathcal{A}/[\mathcal{A},\mathcal{A}]$ and $\widehat{\mathrm{BR}}(\mathcal{A})_n=\mathrm{BR}(\mathcal{A})_n$

The differential

We follow Chapter 4 of [F-Valeri,2509.21232]

Recall: $\widehat{\mathrm{BR}}(\mathcal{A}) = \oplus_{n\geq 0}\widehat{\mathrm{BR}}(\mathcal{A})_n$ with
$\widehat{\mathrm{BR}}(\mathcal{A})_0 = \mathcal{A}_\sharp := \mathcal{A}/[\mathcal{A},\mathcal{A}]$ and $\widehat{\mathrm{BR}}(\mathcal{A})_n = \mathrm{span}_{\Bbbk}\{$all n-brackets$\}$

We denote the double Poisson bracket as $[\![-,-]\!]$

Definition

$\widehat{\mathrm{d}} : \widehat{\mathrm{BR}}(\mathcal{A}) \to \oplus_{n\geq 0} \mathrm{Hom}_{\Bbbk}(\mathcal{A}^{\otimes(n+1)}, \mathcal{A}^{\otimes(n+1)})$ is defined <u>in deg. 0</u> by
$\widehat{\mathrm{d}}(\bar{a}) : \mathcal{A} \to \mathcal{A}, \quad \widehat{\mathrm{d}}(\bar{a}) := -\mathrm{m}\circ [\![a,-]\!]$, ($a \in \mathcal{A}$ lift of $\bar{a} \in \mathcal{A}_\sharp$)

<u>in deg. $n \geq 1$:</u> $\quad \widehat{\mathrm{d}}(\{\!\{-\}\!\})(a_1 \otimes \ldots \otimes a_{n+1})$

$$= \sum_{i=1}^{n+1} (-1)^{ni}\, \sigma^i\ \{\!\{a_{i+1}, \ldots, a_{n+1}, a_1, \ldots, a_{i-2}, [\![a_{i-1}, a_i]\!]\}\!\}_L$$

$$+ \sum_{i=1}^{n+1} (-1)^{ni}\, \sigma^{i-1}\ [\![a_i, \{\!\{a_{i+1}, \ldots, a_{n+1}, a_1, \ldots, a_{i-1}\}\!\}\,]\!]_L$$

Recall: $(a_1 \otimes \cdots \otimes a_{n+1})^\sigma = a_{n+1} \otimes a_1 \otimes \cdots \otimes a_n$

The cohomology

Theorem

Consider the operation $\widehat{\mathrm{d}}$ from the previous definition

1. *For any $n \geq 0$, $\widehat{\mathrm{d}}(\widehat{\mathrm{BR}}(\mathcal{A})_n) \subset \widehat{\mathrm{BR}}(\mathcal{A})_{n+1}$.*
   Hence, we can view $\widehat{\mathrm{d}}$ as a degree $+1$ endomorphism of $\widehat{\mathrm{BR}}(\mathcal{A})$.
2. *The linear map $\widehat{\mathrm{d}}$ is a square-zero differential on $\widehat{\mathrm{BR}}(\mathcal{A})$.*

completed double Poisson cohomology

$$\widehat{\mathrm{dPH}}(\mathcal{A}) = \mathrm{H}(\widehat{\mathrm{BR}}(\mathcal{A}), \widehat{\mathrm{d}})$$

Operations on representation algebras

Theorem

Assume that $[\![-,-]\!] \in \widehat{\mathrm{BR}}(\mathcal{A})_2$ *is a double Poisson bracket, and let* $\{-,-\} = \mathrm{tr}([\![-,-]\!])$ *denote the associated Poisson bracket on* $\mathcal{A}_N$*. Then, there is a morphism of complexes*

$$\mathrm{tr} : (\widehat{\mathrm{BR}}(\mathcal{A}), \widehat{\mathrm{d}}) \longrightarrow (\mathfrak{X}(\mathcal{A}_N), (-1)^{\bullet}\, \delta_{\mathcal{A}_N, \{-,-\}}),$$

which descends to a linear map $\widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE}(\mathcal{A}_N)$ *in cohomology. Furthermore, by* GL_N*-invariance, this descends to a linear map* $\widehat{\mathrm{dPH}}(\mathcal{A}) \to \mathrm{H}_{CE}(\mathcal{A}_N^{\mathrm{GL}_N})$ *in cohomology.*

Proof.

Commutativity of

$$\begin{tikzcd} \widehat{\mathrm{BR}}(\mathcal{A})_{k-1} \arrow[r, "\widehat{\mathrm{d}}"'] \arrow[d, "\mathrm{tr}"'] & \widehat{\mathrm{BR}}(\mathcal{A})_k \arrow[d, "\mathrm{tr}"] \\ \mathfrak{X}^{k-1}(\mathcal{A}_N) \arrow[r, "(-1)^{k-1}\,\delta^{k-1}"] & \mathfrak{X}^k(\mathcal{A}_N) \end{tikzcd}$$

□

2-fold λ-brackets

$(\mathcal{V}, \partial)$ is a differential unital assoc. algebra

Definition ([De Sole-Kac-Valeri,'15])

A 2-fold λ-bracket is a $\Bbbk$-linear map $\{\!\{-_\lambda-\}\!\} : \mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}[\lambda]$ s.t.

1. $\{\!\{a_\lambda b\}\!\} = -\left(|_{x=\partial}\{\!\{b_{-\lambda-x}a\}\!\}^\sigma\right)$ (cyclic skewsymmretry)
2. $\{\!\{a_\lambda bc\}\!\} = (b \otimes 1)\,\{\!\{a_\lambda c\}\!\} + \{\!\{a_\lambda b\}\!\}\,(1 \otimes c)$ (left Leibniz rule)
3. $\{\!\{ad_\lambda b\}\!\} = (1 \otimes |_{x=\partial}a)\,\{\!\{d_{\lambda+x}b\}\!\} + \{\!\{a_{\lambda+x}b\}\!\}\,(|_{x=\partial}d \otimes 1)$ (right Leibniz rule)
4. $\{\!\{\partial a_\lambda b\}\!\} = -\lambda\{\!\{a_\lambda b\}\!\}, \quad \{\!\{a_\lambda \partial b\}\!\} = (\lambda + \partial)\{\!\{a_\lambda b\}\!\}$ (sesquilinearity)

Note: under (1), (2) and (3) are equivalent. So only need (1),(2),(4)

DPVA

$(\mathcal{V},\partial)$ with 2-fold λ-bracket $\{\!\{-_\lambda-\}\!\}:\mathcal{V}^{\otimes 2}\to\mathcal{V}^{\otimes 2}[\lambda]$. We define:

$\{\!\{a_\lambda(b\otimes c)\}\!\}_L := \{\!\{a_\lambda b\}\!\}\otimes c\,,\quad \{\!\{a_\lambda(b\otimes c)\}\!\}_R := b\otimes\{\!\{a_\lambda c\}\!\}\,,$

$\{\!\{(a\otimes b)_\lambda c\}\!\}_L := \{\!\{a_{\lambda+x} c\}\!\}\otimes_1 (|_{x=\partial} b)\,.$ $\quad for\ (x'\otimes x'')\otimes_1 y := x'\otimes y\otimes x''$

Definition (DPVA, [De Sole-Kac-Valeri,'15])

$(\mathcal{V},\partial,\{\!\{-_\lambda-\}\!\})$ is a double Poisson vertex algebra when $\forall a,b,c\in\mathcal{V}$

$$\{\!\{a_\lambda\{\!\{b_\mu c\}\!\}\}\!\}_L-\{\!\{b_\mu\{\!\{a_\lambda c\}\!\}\}\!\}_R-\{\!\{\{\!\{a_\lambda b\}\!\}_{\lambda+\mu} c\}\!\}_L=0\,.$$

DPVA

$(\mathcal{V}, \partial)$ with 2-fold λ-bracket $\{\!\{-_\lambda-\}\!\} : \mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}[\lambda]$. We define:
$\{\!\{a_\lambda(b\otimes c)\}\!\}_L := \{\!\{a_\lambda b\}\!\} \otimes c\ , \quad \{\!\{a_\lambda(b\otimes c)\}\!\}_R := b \otimes \{\!\{a_\lambda c\}\!\}\ ,$
$\{\!\{(a\otimes b)_\lambda c\}\!\}_L := \{\!\{a_{\lambda+x} c\}\!\} \otimes_1 (|_{x=\partial} b)\ .$ $\quad$ *for* $(x'\otimes x'')\otimes_1 y := x'\otimes y\otimes x''$

Definition (DPVA, [De Sole-Kac-Valeri,'15])

$(\mathcal{V}, \partial, \{\!\{-_\lambda-\}\!\})$ is a double Poisson vertex algebra when $\forall a,b,c \in \mathcal{V}$

$$\{\!\{a_\lambda\{\!\{b_\mu c\}\!\}\}\!\}_L - \{\!\{b_\mu\{\!\{a_\lambda c\}\!\}\}\!\}_R - \{\!\{\{\!\{a_\lambda b\}\!\}_{\lambda+\mu} c\}\!\}_L = 0\,.$$

Example ([DSKV,'15])

For $\mathcal{V}_1 = \Bbbk\langle u_j \mid j \in \mathbb{Z}_{\geq 0}\rangle$, $\partial : u_j \mapsto u_{j+1}$.
Unique DPVA structure such that $\{\!\{u_{0\,\lambda} u_0\}\!\} = 1\otimes u_0 - u_0 \otimes 1 + (1\otimes 1)\,\lambda$

Example ([Bozec-F-Moreau,'25])

If $(\mathcal{A}, \{\!\{-_\lambda-\}\!\})$ is a DPA, then its jet algebra $J_\infty\mathcal{A}$ is a DPVA for the unique 2-fold λ-bracket determined by: $\forall a,b \in \mathcal{A} \subset J_\infty\mathcal{A}$, $\{\!\{a_\lambda b\}\!\} = \{\!\{a,b\}\!\}$.

Plan for the talk

1. Motivation
2. PVA cohomologies
3. dPA and dPVA
4. **dPVA cohomology**

Big picture about DPVA cohomologies

We will follow Chapters 9,10 & 13 of [F-Valeri,2509.21232]

- $\widetilde{\Gamma}(\mathcal{V})$ space of *basic (NC) cochains*
  $\rightsquigarrow$ basic PVA cohomology $\mathrm{dH}_{\mathrm{bas}}(\mathcal{V})$
- $\Gamma(\mathcal{V}) = \widetilde{\Gamma}(\mathcal{V})/\big([\widetilde{\Gamma}(\mathcal{V}),\widetilde{\Gamma}(\mathcal{V})] + \partial\widetilde{\Gamma}(\mathcal{V})\big)$
  $\rightsquigarrow$ reduced PVA cohomology $\mathrm{dH}_{\mathrm{red}}(\mathcal{V})$
- $C^n(\mathcal{V})$ space of n-fold λ-*brackets*
  $\rightsquigarrow$ variational PVA cohomology $\mathrm{dP_vH}(\mathcal{V})$

$$\begin{tikzcd}
\boxed{\begin{matrix}\mathrm{dH}_{\mathrm{bas}}(\mathcal{V}) \longrightarrow \mathrm{dH}_{\mathrm{red}}(\mathcal{V}) \\ \qquad\qquad\quad \big\downarrow \\ \qquad\qquad\quad \mathrm{dP_vH}(\mathcal{V})\end{matrix}} \arrow[r, dashed] & \begin{matrix}\mathrm{H}_{\mathrm{bas}}(\mathcal{V}_N) \longrightarrow \mathrm{H}_{\mathrm{red}}(\mathcal{V}_N) \\ \qquad\qquad\quad \big\downarrow \\ \qquad\qquad\quad \mathrm{P_vH}(\mathcal{V}_N)\end{matrix}
\end{tikzcd}$$

Use functors $(\mathcal{V},\partial) \longrightarrow (\mathcal{V}_N,\partial)$ for differential algebras $\big(\, \partial(a_{ij}) := (\partial a)_{ij} \,\big)$

Algebras of NC differential polynomials: vertical maps are isomorphisms

NC basic cochains $\widetilde{\Gamma}(\mathcal{V})$

degree 0: $\widetilde{\Gamma}^0(\mathcal{V}) = \mathcal{V}$

degree $n \geq 1$: $X \in \widetilde{\Gamma}^n(\mathcal{V})$ is a basic n-cochain when

$$X : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes(n+1)}[\lambda_1, \ldots, \lambda_n], \quad a_1 \otimes \cdots \otimes a_n \mapsto X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_n),$$

satisfies ~~skewsymmetry~~ Leibniz rules + sesquilinearity as

$$X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_{i-1}, bc, a_{i+1}, \ldots, a_n)$$
$$= (|_{x=\partial}b) \star_i X_{\lambda_1,\ldots,\lambda_{i-1},\lambda_i+x,\lambda_{i+1},\ldots\lambda_n}(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n)$$
$$+X_{\lambda_1,\ldots,\lambda_{i-1},\lambda_i+x,\lambda_{i+1},\ldots\lambda_n}(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \star_{n+1-i} (|_{x=\partial}c);$$
$$X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_{i-1}, \partial a_i, a_{i+1}, \ldots, a_n)$$
$$= -\lambda_i X_{\lambda_1,\ldots,\lambda_n}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n).$$

$\star_j$ says how many jumps over tensor products to perform:

$$b \star_i (a_1 \otimes \cdots \otimes a_{n+1}) \star_{n+1-i} c = a_1 \otimes \cdots \otimes a_i c \otimes b a_{i+1} \otimes \cdots \otimes a_{n+1}$$

NC basic cochains $\widetilde{\Gamma}(\mathcal{V})$

degree 0: $\widetilde{\Gamma}^0(\mathcal{V}) = \mathcal{V}$

degree $n \geq 1$: $X \in \widetilde{\Gamma}^n(\mathcal{V})$ is a basic n-cochain when

$$X : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes(n+1)}[\lambda_1, \dots, \lambda_n], \quad a_1 \otimes \cdots \otimes a_n \mapsto X_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_n),$$

satisfies ~~skewsymmetry~~ Leibniz rules + sesquilinearity as

$$X_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_{i-1}, bc, a_{i+1}, \dots, a_n)$$
$$= (|_{x=\partial} b) \star_i X_{\lambda_1,\dots,\lambda_{i-1},\lambda_i+x,\lambda_{i+1},\dots\lambda_n}(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n)$$
$$+ X_{\lambda_1,\dots,\lambda_{i-1},\lambda_i+x,\lambda_{i+1},\dots\lambda_n}(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \star_{n+1-i} (|_{x=\partial} c);$$
$$X_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_{i-1}, \partial a_i, a_{i+1}, \dots, a_n)$$
$$= -\lambda_i X_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n).$$

$\star_j$ says how many jumps over tensor products to perform:

$$b \star_i (a_1 \otimes \cdots \otimes a_{n+1}) \star_{n+1-i} c = a_1 \otimes \cdots \otimes a_i c \otimes b a_{i+1} \otimes \cdots \otimes a_{n+1}$$

- $\exists$ (graded) concatenation product $\widetilde{\Gamma}^n(\mathcal{V}) \otimes \widetilde{\Gamma}^\ell(\mathcal{V}) \to \widetilde{\Gamma}^{n+\ell}(\mathcal{V})$
- $\partial \curvearrowright \widetilde{\Gamma}^n(\mathcal{V})$ by $(\partial X)_{\underline{\lambda}}(\underline{a}) := (\lambda_1 + \cdots + \lambda_m + \partial) X_{\underline{\lambda}}(\underline{a})$

  $\partial(a_1 \otimes \cdots \otimes a_{n+1}) := \sum_{j=1}^{n+1} a_1 \otimes \cdots \otimes \partial(a_j) \otimes \cdots \otimes a_{n+1}$

The differential on $\widetilde{\Gamma}(\mathcal{V})$

Assume that $(\mathcal{V}, [\![-_\lambda-]\!])$ is a dPVA.

Definition

For every $X \in \widetilde{\Gamma}^n(\mathcal{V})$ let $\tilde{\delta}(X) : \mathcal{V}^{\otimes(n+1)} \to \mathcal{V}^{\otimes(n+2)}[\lambda_1, \dots, \lambda_{n+1}]$ be

$$\tilde{\delta}(X)_{\lambda_1,\dots,\lambda_{n+1}}(a_1, \dots, a_{n+1}) = \sum_{s=1}^{n+1} (-1)^{s+1} [\![{a_s}_{\lambda_s} X_{\lambda_1, \overset{s}{\check{\dots}}, \lambda_{n+1}}(a_1, \overset{s}{\check{\dots}}, a_{n+1})]\!]_{(s)}$$

$$+ \sum_{s=1}^{n} (-1)^s X^{(s)}_{\lambda_1,\dots,\lambda_{s-1},\lambda_s+\lambda_{s+1},\lambda_{s+2},\dots,\lambda_{n+1}}(a_1, \dots, a_{s-1}, [\![{a_s}_{\lambda_s} a_{s+1}]\!], a_{s+2}, \dots, a_{n+1})$$

$[\![a_\lambda (b_1 \otimes \cdots \otimes b_{n+1})]\!]_{(s)} = b_1 \otimes \cdots \otimes [\![a_\lambda b_s]\!] \otimes \cdots \otimes b_{n+1}$

$X^{(s)}_{\underline{\mu}}(b_1, \dots, b_s \otimes C, \dots, b_{n+1}) = X_{\mu_1,\dots,\mu_s+x,\dots,\mu_{n+1}}(b_1, \dots, b_s, \dots, b_{n+1}) \otimes_{n+1-s} |_{x=\partial C}$

$\looparrowright$ replace μ_s by $\mu_s + \partial$ acting on C which goes in tensor position $s+1$

The differential on $\widetilde{\Gamma}(\mathcal{V})$

Assume that $(\mathcal{V}, [\![-_\lambda-]\!])$ is a dPVA.

Definition

For every $X \in \widetilde{\Gamma}^n(\mathcal{V})$ let $\tilde{\delta}(X) : \mathcal{V}^{\otimes(n+1)} \to \mathcal{V}^{\otimes(n+2)}[\lambda_1, \dots, \lambda_{n+1}]$ be

$$\tilde{\delta}(X)_{\lambda_1,\dots,\lambda_{n+1}}(a_1,\dots,a_{n+1}) = \sum_{s=1}^{n+1} (-1)^{s+1} [\![a_s{}_{\lambda_s} X_{\lambda_1,\overset{s}{\check{\dots}},\lambda_{n+1}}(a_1,\overset{s}{\check{\dots}},a_{n+1})]\!]_{(s)}$$

$$+ \sum_{s=1}^{n} (-1)^s X^{(s)}_{\lambda_1,\dots,\lambda_{s-1},\lambda_s+\lambda_{s+1},\lambda_{s+2},\dots,\lambda_{n+1}}(a_1,\dots,a_{s-1},[\![a_s{}_{\lambda_s} a_{s+1}]\!],a_{s+2},\dots,a_{n+1})$$

$[\![a_\lambda(b_1 \otimes \cdots \otimes b_{n+1})]\!]_{(s)} = b_1 \otimes \cdots \otimes [\![a_\lambda b_s]\!] \otimes \cdots \otimes b_{n+1}$

$X^{(s)}_{\underline{\mu}}(b_1,\dots,b_s \otimes C,\dots,b_{n+1}) = X_{\mu_1,\dots,\mu_s+x,\dots,\mu_{n+1}}(b_1,\dots,b_s,\dots,b_{n+1}) \otimes_{n+1-s} |_{x=\partial C}$

$\looparrowright$ replace μ_s by $\mu_s + \partial$ acting on C which goes in tensor position $s+1$

Example ($n = 0, 1$)

$\tilde{\delta}(c) = [\![-_\lambda c]\!]$; $\tilde{\delta}(X)_{\lambda,\mu}(a,b) = [\![a_\lambda X_\mu(b)]\!]_{(1)} - [\![b_\mu X_\lambda(a)]\!]_{(2)} - X^{(1)}_{\lambda+\mu}([\![a_\lambda b]\!])$

Then $\tilde{\delta}^2(c)_{\lambda,\mu}(a,b) = [\![a_\lambda [\![b_\mu c]\!]]\!]_L - [\![b_\mu [\![a_\lambda c]\!]]\!]_R - [\![[\![a_\lambda c]\!]_{\lambda+\mu} c]\!]_L$

The basic dPVA cohomology

Theorem

Let $\mathcal{V}$ be a PVA and let $\widetilde{\Gamma}(\mathcal{V})$ be the (super)space of basic cochains.

- (a) *The linear map $\tilde{\delta}$ gives a well-defined map $\tilde{\delta} : \widetilde{\Gamma}^n(\mathcal{V}) \to \widetilde{\Gamma}^{n+1}(\mathcal{V})$. Moreover, $\tilde{\delta}^2 = 0$.*
- (b) *$\tilde{\delta}$ commutes with the action of ∂ on $\widetilde{\Gamma}(V)$.*
- (c) *$\tilde{\delta}$ acts as an odd derivation, in particular it preserves the subspace of graded commutators $[\widetilde{\Gamma}(\mathcal{V}), \widetilde{\Gamma}(\mathcal{V})]$.*

basic dPVA cohomology

$$\mathrm{dH}_{\mathrm{bas}}(\mathcal{V}) = \mathrm{H}(\widetilde{\Gamma}(\mathcal{V}), \tilde{\delta})$$

The reduced dPVA cohomology

By the theorem, we get a subcomplex

$$(\partial\widetilde{\Gamma}(\mathcal{V}) + [\partial\widetilde{\Gamma}(\mathcal{V}), \partial\widetilde{\Gamma}(\mathcal{V})], \tilde{\delta}) \subset (\widetilde{\Gamma}(\mathcal{V}), \tilde{\delta})$$

Then, form the complex $(\Gamma(\mathcal{V}), \delta)$ where

$$\Gamma(\mathcal{V}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma^n(\mathcal{V}), \quad \Gamma^n(\mathcal{V}) = \widetilde{\Gamma}^n(\mathcal{V}) / \big(\partial\widetilde{\Gamma}^n(\mathcal{V}) + [\partial\widetilde{\Gamma}(\mathcal{V}), \partial\widetilde{\Gamma}(\mathcal{V})]\big)$$

and $\delta : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ is induced by $\tilde{\delta}$.

Reduced dPVA cohomology

$$\mathrm{dH}_{\mathrm{red}}(\mathcal{V}) = \mathrm{H}(\Gamma(\mathcal{V}), \delta)$$

multi-fold λ-brackets $C(\mathcal{V})$

<u>degree 0:</u> $C^0(\mathcal{V}) = \mathcal{V}_\sharp := \mathcal{V}/(\partial\mathcal{V} + [\mathcal{V},\mathcal{V}])$

<u>degree $n \geq 1$:</u> A n-fold λ-bracket is

$$\{\!\{-_{\lambda_1} - \cdots -_{\lambda_{n-1}} -\}\!\} : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n}[\lambda_1, \ldots, \lambda_{n-1}]$$

that satisfies cyclic skewsymmetry + Leibniz rules + sesquilinearity as

$$\{\!\{a_{1\,\lambda_1} \ldots a_{n-1\,\lambda_{n-1}} a_n\}\!\} = (-1)^{n+1}\left(\big|_{\lambda_n=\lambda_n^\dagger} \{\!\{a_{2\,\lambda_2} \ldots a_{n\,\lambda_n} a_1\}\!\}^\sigma\right);$$

$$\{\!\{a_{1\,\lambda_1} \cdots {}_{\lambda_{i-1}} bc_{\lambda_i} \ldots a_{n-1\,\lambda_{n-1}} a_n\}\!\}$$
$$\overset{1\leqslant i\leqslant n-1}{=} (|_{x=\partial} b) \star_i \{\!\{a_{1\,\lambda_1} \cdots {}_{\lambda_{i-1}} c_{\lambda_i+x} \ldots a_{n-1\,\lambda_{n-1}} a_n\}\!\}$$
$$+\{\!\{a_{1\,\lambda_1} \cdots {}_{\lambda_{i-1}} b_{\lambda_i+x} \ldots a_{n-1\,\lambda_{n-1}} a_n\}\!\} \star_{n-i} (|_{x=\partial} c);$$

$$\{\!\{a_{1\,\lambda_1} \cdots {}_{\lambda_{i-1}} (\partial a_i)_{\lambda_i} \cdots {}_{\lambda_{n-1}} a_n\}\!\} \overset{1\leqslant i\leqslant n-1}{=} -\lambda_i \{\!\{a_{1\,\lambda_1} \cdots {}_{\lambda_{i-1}} a_{i\,\lambda_i} \cdots {}_{\lambda_{n-1}} a_n\}\!\}$$
$$\{\!\{a_{1\,\lambda_1} \cdots a_{n-1\,\lambda_{n-1}} (\partial a_n)\}\!\} = -\lambda_n^\dagger \{\!\{a_{1\,\lambda_1} \cdots a_{n-1\,\lambda_{n-1}} a_n\}\!\}.$$

where $\lambda_n^\dagger := -\lambda_1 - \cdots - \lambda_{n-1} - \partial$

The differential on $C(\mathcal{V})$

Assume that $(\mathcal{V}, [\![-_\lambda-]\!])$ is a dPVA.

Definition

For $\int f \in C^0(\mathcal{V}) = \mathcal{V}_\sharp$, $\mathrm{d}(\int f) = -\,\mathrm{m} \circ [\![f_\lambda -]\!]|_{\lambda=0} : \mathcal{V} \to \mathcal{V}$;
For $Q = \{\!\{-_{\lambda_1} - \cdots -_{\lambda_{n-1}} -\}\!\} \in C^n(\mathcal{V})$, let
$\mathrm{d}(Q) : \mathcal{V}^{\otimes(n+1)} \to \mathcal{V}^{\otimes(n+1)}[\lambda_1, \dots, \lambda_n]$ be

$$\begin{aligned}
&\mathrm{d}(Q)_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_{n+1}) \\
&= \sum_{s=1}^{n} (-1)^{n+s+1} [\![a_{s\,\lambda_s} \{\!\{ a_{1\,\lambda_1} \overset{s}{\check{\cdots}} a_{n\,\lambda_n} a_{n+1} \}\!\}]\!]_{(s)} \\
&- [\![\{\!\{ a_{1\,\lambda_1} \dots a_{n-1\,\lambda_{n-1}} a_n \}\!\}_{\lambda_1+\cdots+\lambda_n} a_{n+1}]\!]_L \\
&+ \sum_{s=1}^{n} (-1)^{n+s} \{\!\{ a_{1\,\lambda_1} \dots a_{s-1\,\lambda_{s-1}} [\![a_{s\,\lambda_s} a_{s+1}]\!]_{\lambda_s+\lambda_{s+1}} a_{s+2} \dots {}_{\lambda_n} a_{n+1} \}\!\}_L \\
&+ (-1)^{n+1} \{\!\{ a_{2\,\lambda_2} \dots a_{n\,\lambda_n} [\![a_{1\,\lambda_1} a_{n+1}]\!] \}\!\}_R \,.
\end{aligned}$$

Notation for first and third terms: as in $\widetilde{\delta}(X)$; for fourth: $\{\!\{B_\mu c' \otimes c''\}\!\}_R = c' \otimes \{\!\{B_\mu c''\}\!\}$
for second: $[\![(b_1 \otimes \cdots \otimes b_n)_\mu a]\!]_L = [\![b_{1\,\mu+x} a]\!] \otimes_1 (|_{x=\partial} b_2 \otimes \cdots \otimes b_n)$

The differential on $C(\mathcal{V})$ (example)

Example

Let $D \in C^1(\mathcal{V}) \rightsquigarrow$ derivation commuting with ∂.

$$\mathrm{d}(D)_\lambda(a,b) = D([\![a_\lambda b]\!]) - [\![D(a)_\lambda b]\!] - [\![a_\lambda D(b)]\!]\,,$$

where on first term of RHS $D(c' \otimes c'') := D(c') \otimes c'' + c' \otimes D(c'')$.
One has $\mathrm{d}(D) \in C^2(\mathcal{V})$ because one can check:

$$\mathrm{d}(D)_\lambda(a,bc) = \mathrm{d}(D)_\lambda(a,b)\,c + b\,\mathrm{d}(D)_\lambda(a,c)\,,$$
$$\mathrm{d}(D)_\lambda(a,b) = -|_{x=\partial}\,\mathrm{d}(D)_{-\lambda-x}(b,a)^\sigma\,.$$

Finally, $\mathrm{d}^2 : C^0(\mathcal{V}) \to C^2(\mathcal{V})$ is the trivial map:

$$\begin{aligned}\mathrm{d}^2(\textstyle\int a)_\mu(b,c) &= -[a,[\![b_\mu c]\!]] + [\![[a,b]_\mu c]\!] + [\![b_\mu [a,c]]\!] \\ = -\,&\left([\![a_\lambda [\![b_\mu c]\!]]\!]_L - [\![b_\mu [\![a_\lambda c]\!]]\!]_R - [\![[\![a_\lambda b]\!]_{\lambda+\mu} c]\!]_L\right)|_{\lambda=0} \\ +\,&\left([\![b_\mu [\![a_\lambda c]\!]]\!]_L - [\![a_\lambda [\![b_\mu c]\!]]\!]_R - [\![[\![b_\mu a]\!]_{\lambda+\mu} c]\!]_L\right)|_{\lambda=0} = 0\end{aligned}$$

The variational dPVA cohomology

Theorem

Let $\mathcal{V}$ be a PVA, $C(\mathcal{V})$ be the space of multifold λ-brackets.

Then, the linear map d gives a well-defined map $\mathrm{d} : C^n(\mathcal{V}) \to C^{n+1}(\mathcal{V})$.

Moreover, $\mathrm{d}^2 = 0$.

variational dPVA cohomology

$$\mathrm{dP_vH}(\mathcal{V}) = \mathrm{H}(C(\mathcal{V}), \mathrm{d})$$

The variational dPVA cohomology

Theorem

Let $\mathcal{V}$ be a PVA, $C(\mathcal{V})$ be the space of multifold λ-brackets.

Then, the linear map d gives a well-defined map $\mathrm{d} : C^n(\mathcal{V}) \to C^{n+1}(\mathcal{V})$.

Moreover, $\mathrm{d}^2 = 0$.

variational dPVA cohomology

$$\mathrm{dP_vH}(\mathcal{V}) = \mathrm{H}(C(\mathcal{V}), \mathrm{d})$$

$$\mathrm{dP_vH}^0(\mathcal{V}) = \{\textstyle\int a \in \mathcal{V}_\sharp \mid [\![\textstyle\int a, b]\!] = 0\,,\ \text{for every } b \in \mathcal{V}\} = \mathrm{Cas}(\mathcal{V})$$

$$\mathrm{dP_vH}^1(\mathcal{V}) = \frac{\{\text{dPVA-derivations of } \mathcal{V}\}}{\{\text{inner dPVA-derivations of } \mathcal{V}\}}$$

$$= \frac{\{D \in \mathrm{Der}(\mathcal{V}) | D\circ\partial = \partial\circ D,\ D([\![a_\lambda b]\!]) = [\![D(a)_\lambda b]\!] + [\![a_\lambda D(b)]\!]\}}{\{[\![\int f, -]\!] | f \in \mathcal{V}\}}$$

$\mathrm{dP_vH}^2(\mathcal{V})$ controls formal deformations. See also [Zheng-Tan, private com.]

The map $\Gamma(\mathcal{V}) \to C(\mathcal{V})$

For $f \in \mathcal{V} = \widetilde{\Gamma}^0(\mathcal{V})$, let $\widetilde{\mathrm{P}}_0(f) = \int f$.
For $X \in \widetilde{\Gamma}^n(\mathcal{V})$, $n \geq 1$, define $\widetilde{\mathrm{P}}_n(X) : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n}[\lambda_1, \ldots, \lambda_{n-1}]$ by

$$\boxed{\widetilde{\mathrm{P}}_n(X) = \sum_{s=0}^{n-1} \frac{(-1)^{s(n-s)}}{n} \Big|_{\lambda_n = \lambda_n^\dagger} \mathrm{m}_{(s+1,s+2)} \circ \sigma^{s+1} \circ X_{\lambda_{\sigma^s(1)},\ldots,\lambda_{\sigma^s(n)}} \circ \sigma^{-s}}$$

Proposition

1. *Let* $X \in \widetilde{\Gamma}^n(\mathcal{V})$. *Then* $\widetilde{\mathrm{P}}_n(X) \in C^n(\mathcal{V})$.
2. *Let* $X \in \widetilde{\Gamma}^n(\mathcal{V})$. *Then* $\widetilde{\mathrm{P}}_n(\partial X) = 0$.
3. *Let* $X \in \widetilde{\Gamma}^m(\mathcal{V})$ *and* $Y \in \widetilde{\Gamma}^n(\mathcal{V})$. *Then* $\widetilde{\mathrm{P}}_{m+n}([X, Y]) = 0$.

The map $\Gamma(\mathcal{V}) \to C(\mathcal{V})$

For $f \in \mathcal{V} = \widetilde{\Gamma}^0(\mathcal{V})$, let $\widetilde{\mathrm{P}}_0(f) = \int f$.
For $X \in \widetilde{\Gamma}^n(\mathcal{V})$, $n \geq 1$, define $\widetilde{\mathrm{P}}_n(X) : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n}[\lambda_1, \dots, \lambda_{n-1}]$ by

$$\widetilde{\mathrm{P}}_n(X) = \sum_{s=0}^{n-1} \frac{(-1)^{s(n-s)}}{n} \Big|_{\lambda_n = \lambda_n^\dagger} \mathrm{m}_{(s+1,s+2)} \circ \sigma^{s+1} \circ X_{\lambda_{\sigma^s(1)}, \dots, \lambda_{\sigma^s(n)}} \circ \sigma^{-s}$$

Proposition

1. *Let* $X \in \widetilde{\Gamma}^n(\mathcal{V})$. *Then* $\widetilde{\mathrm{P}}_n(X) \in C^n(\mathcal{V})$.
2. *Let* $X \in \widetilde{\Gamma}^n(\mathcal{V})$. *Then* $\widetilde{\mathrm{P}}_n(\partial X) = 0$.
3. *Let* $X \in \widetilde{\Gamma}^m(\mathcal{V})$ *and* $Y \in \widetilde{\Gamma}^n(\mathcal{V})$. *Then* $\widetilde{\mathrm{P}}_{m+n}([X, Y]) = 0$.

(2)+(3): the maps factor through $\mathrm{P} : \Gamma(\mathcal{V}) \to C(\mathcal{V})$

Proposition

For every $X \in \widetilde{\Gamma}^n(\mathcal{V})$, $n \in \mathbb{Z}_{\geq 0}$, *we have*

$$(n+1)\widetilde{\mathrm{P}}_{n+1}(\tilde{\delta}(X)) = (n + \delta_{n,0})(-1)^n \,\mathrm{d}(\widetilde{\mathrm{P}}_n(X))$$

$$\begin{tikzcd}
\Gamma^0(\mathcal{V}) \arrow[r, "\delta"] \arrow[d, "="'] & \Gamma^1(\mathcal{V}) \arrow[d, "\mathrm{P}_1"] \\
C^0(\mathcal{V}) \arrow[r, "\mathrm{d}"] & C^1(\mathcal{V})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma^n(\mathcal{V}) \arrow[r, "\delta"] \arrow[d, "n\,\mathrm{P}_n"'] & \Gamma^{n+1}(\mathcal{V}) \arrow[d, "(n+1)\,\mathrm{P}_{n+1}"] \\
C^n(\mathcal{V}) \arrow[r, "(-1)^n\,\mathrm{d}"] & C^{n+1}(\mathcal{V})
\end{tikzcd}$$

The map $\Gamma(\mathcal{V}) \to C(\mathcal{V})$ in cohomology

Proposition

For every $X \in \widetilde{\Gamma}^n(\mathcal{V})$, $n \in \mathbb{Z}_{\geq 0}$, *we have*

$$(n+1)\widetilde{\mathrm{P}}_{n+1}(\tilde{\delta}(X)) = (n+\delta_{n,0})(-1)^n \,\mathrm{d}(\widetilde{\mathrm{P}}_n(X))$$

$$\begin{tikzcd} \Gamma^0(\mathcal{V}) \arrow[r, "\delta"] \arrow[d, "="'] & \Gamma^1(\mathcal{V}) \arrow[d, "\mathrm{P}_1"] \\ C^0(\mathcal{V}) \arrow[r, "\mathrm{d}"] & C^1(\mathcal{V}) \end{tikzcd} \qquad \begin{tikzcd} \Gamma^n(\mathcal{V}) \arrow[r, "\delta"] \arrow[d, "n\,\mathrm{P}_n"'] & \Gamma^{n+1}(\mathcal{V}) \arrow[d, "(n+1)\,\mathrm{P}_{n+1}"] \\ C^n(\mathcal{V}) \arrow[r, "(-1)^n\,\mathrm{d}"] & C^{n+1}(\mathcal{V}) \end{tikzcd}$$

Theorem

$\forall n \in \mathbb{Z}_{\geq 0}$ *the maps* $\mathrm{P}_n : \Gamma^n(\mathcal{V}) \to C^n(\mathcal{V})$ *give linear maps in cohomology*

$$\mathrm{dH}^n_{\mathrm{red}}(\mathcal{V}) \to \mathrm{dP_vH}^n(\mathcal{V})\,.$$

In particular, $\mathrm{dH}^0_{\mathrm{red}}(\mathcal{V}) \simeq \mathrm{dP_vH}^0(\mathcal{V})$ *provided that* P_1 *is injective.*

Calculations

Let $\mathcal{V} := \mathcal{R}_\ell = \Bbbk\langle u_i^{(p)} \mid i \in I, p \in \mathbb{Z}_{\geq 0}\rangle$
$\rightsquigarrow$ algebra of NC differential polynomials in ℓ variables $u_1, \dots, u_\ell$.

Let $M \in \mathbb{Z}_{\geq 0}$ and let $K = (K_{ij})_{i,j=1}^{\ell} \in \mathrm{GL}_\ell(\Bbbk)$ be a symmetric matrix, if M is odd, or a skew-symmetric matrix, if M is even.

Get a dPVA structure on $\mathcal{V}$ from $\{\!\{u_i{}_\lambda u_j\}\!\} = K_{ji}(1 \otimes 1)\,\lambda^M$.

Theorem

When $M = 0$, $\mathrm{dP_vH}^n(\mathcal{V}) = \delta_{n0}\Bbbk$.

When $M \geq 1$, $\mathrm{dP_vH}^n(\mathcal{V})$ can be characterized more or less explicitly.
This is very tedious, see [F-V, Chapter 12] for details...

Calculations ($M = 1$)

$\mathcal{V} := \mathcal{R}_\ell$, $K = (K_{ij})_{i,j=1}^{\ell} \in \mathrm{GL}_\ell(\Bbbk)$ symmetric

$\rightsquigarrow \{\!\{u_{i\lambda} u_j\}\!\} = K_{ji}(1 \otimes 1)\,\lambda \qquad$ for $i, j = 1, \dots, \ell$

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Example ($\mathrm{dP_vH}^1(\mathcal{V}) \simeq \Bbbk^\ell \oplus \mathfrak{so}_\ell$)

For $a \in \Bbbk^\ell$, $b \in V_2 \simeq \mathfrak{so}_\ell$, setting $w_j := \sum_h (K^{-1})_{jh} u_h$,
$D_{a,b} \in \mathrm{Vect}(\mathcal{V})^\partial$ defined on generators by $D_a(u_i) = a_i + \sum_{j \in I} b_{ji} w_j$

Calculations ($M = 1$)

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Example ($\mathrm{dP_vH}^2(\mathcal{V})$: First-order formal deformation)

For $b \in V_2 \simeq \mathfrak{so}_\ell$, $c \in V_3$ $(c_{ijk} = c_{jki})$
$\{\!\{u_{i\,\lambda} u_j\}\!\}_\epsilon^{b,c} = K_{ji}(1 \otimes 1)\lambda + \epsilon\, b_{ji}(1 \otimes 1) + \epsilon \sum_h \left(c_{hji} w_h \otimes 1 - 1 \otimes c_{hij} w_h\right)$

New dPVA structures (that are like affine PVAs $\mathcal{V}(\mathfrak{g})$)
$\{\!\{x_\lambda y\}\!\} = (x|y)_K\,(1 \otimes 1)\lambda + \langle x|y\rangle_b\,(1 \otimes 1) + (x \circ_c y) \otimes 1 - 1 \otimes (y \circ_c x)$
$(\cdot|\cdot)_K$ symmetric, $\langle \cdot|\cdot\rangle_b$ skewsymmetric, $\circ_c$ associative product
with $(x|y \circ_c z)_K = (x \circ_c y|z)_K$, $\langle x|y \circ_c z\rangle_b + \langle y|z \circ_c x\rangle_b + \langle z|x \circ_c y\rangle_b = 0$

Relation to the Kontsevich-Rosenberg principle

$\mathcal{V}_N$ generated by symbols a_{ij}, $\forall a \in \mathcal{V}$, $1 \leq i, j \leq N$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.
∂ lifts to $\mathcal{V}_N$ as follows on generators: $\partial(a_{ij}) := \partial(a)_{ij}$

Theorem ([De Sole-Kac-Valeri,'15])

If $(\mathcal{V}, [\![-_\lambda-]\!])$ is a dPVA then, for any $N \geq 1$,
$\mathcal{V}_N$ is a PVA for the λ-bracket $\{-_\lambda-\}$ uniquely defined by

$$\{a_{ij}{}_\lambda b_{kl}\} = \sum_{r \geq 0} {c^{r}}'_{kj}\, {c^{r}}''_{il}\, \lambda^r, \quad \textit{where } [\![a_\lambda b]\!] := \sum_{r \geq 0} ({c^r}' \otimes {c^r}'')\, \lambda^r$$

Proof.

$$\begin{aligned}
&\{a_{ij}{}_\lambda \{b_{kl}{}_\mu c_{uv}\}\} - \{b_{kl}{}_\mu \{a_{ij}{}_\lambda c_{uv}\}\} - \{\{a_{ij}{}_\lambda b_{kl}\}_{\lambda+\mu} c_{uv}\} \\
&= + \left([\![a_\lambda [\![b_\mu c]\!]]\!]_L - [\![b_\mu [\![a_\lambda c]\!]]\!]_R - [\![[\![a_\lambda b]\!]_{\lambda+\mu} c]\!]_L\right)_{uj,il,kv} \\
&\quad - \left([\![b_\mu [\![a_\lambda c]\!]]\!]_L - [\![a_\lambda [\![b_\mu c]\!]]\!]_R - [\![[\![b_\mu a]\!]_{\lambda+\mu} c]\!]_L\right)_{kj,iv,ul}
\end{aligned}$$

$\square$

The trace map $\mathrm{tr} : C(\mathcal{V}) \to C(\mathcal{V}_N)$

$\underline{n=0}$: one has $\mathrm{tr} : \mathcal{V} \to \mathcal{V}_N$ given by $\mathrm{tr}(f) = \sum_{i=1}^N f_{ii}$
which descends to $\mathrm{tr} : C^0(\mathcal{V}) \to C^0(\mathcal{V}_N)$, $\mathrm{tr}(\int f) = \int (\mathrm{tr}\, f)$

$\underline{n=1}$: one has $\mathrm{tr} : C^1(\mathcal{V}) \to \mathrm{Der}(\mathcal{V}_N)$ given by $\mathrm{tr}(D)(a_{ij}) = D(a)_{ij}$
D commutes with $\partial \rightsquigarrow \mathrm{tr}(D) \in C^1(\mathcal{V}_N)$

$\underline{n=2}$: one has $\mathrm{tr} : C^2(\mathcal{V}) \to C^2(\mathcal{V}_N)$ given by

$$\mathrm{tr}(\{\!\{-_\lambda-\}\!\})(a_{ij}, b_{kl}) := \sum_{r\geq 0} {c^{r}}'_{kj}\, {c^{r}}''_{il}\, \lambda^r , \quad \text{for } \{\!\{a_\lambda b\}\!\} := \sum_{r\geq 0} ({c^r}' \otimes {c^r}'')\, \lambda^r$$

(cf. previous theorem)

$\underline{n \geqslant 3}$: explicit "higher formulas"

Operations on representation algebras

Theorem

Assume that $(\mathcal{V}, [\![-_\lambda-]\!])$ *is a dPVA and let* $(\mathcal{V}_N, [-_\lambda-])$ *be the corresponding PVA obtained through the previous theorem.*
Then, there is a morphism of complexes

$$\mathrm{tr} : (C(\mathcal{V}), \mathrm{d}) \longrightarrow (C(\mathcal{V}_N), (-1)^\bullet \, \mathrm{d}_N),$$

which descends to a linear map $\mathrm{dP_vH}(\mathcal{V}) \to \mathrm{P_vH}(\mathcal{V}_N)$ *in cohomology.*
Furthermore, by GL_N*-invariance, this descends to a linear map* $\mathrm{dP_vH}(\mathcal{V}) \to \mathrm{P_vH}(\mathcal{V}_N^G)$ *in cohomology.*

Operations on representation algebras

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Furthermore, by GL_N*-invariance, this descends to a linear map* $\mathrm{dP_vH}(\mathcal{V}) \to \mathrm{P_vH}(\mathcal{V}_N^G)$ *in cohomology.*

In fact, this can be related to the theorem for dPA through:

- the jet construction (d)PA→(d)PVA;
- the quotient construction (d)PVA→(d)PA.

See [F-V, Chapters 14-15]

Thank you for listening !

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