Representations of double quasi-Poisson algebras in types B,C,D

(joint work with S. Arthamonov)

Maxime Fairon

Institut de Mathématiques de Bourgogne (IMB)
Université Bourgogne Europe & CNRS

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Why should you care?

- (Geometry) Many nice Poisson moduli spaces (e.g. quiver varieties) come from representations of a NC algebra
- (Integrable System) The dynamics of many integrable systems takes place on moduli spaces of representations of a NC algebra

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$$\begin{array}{ccc}
\begin{array}{c}\text{Algebra } \mathcal{A} \\ \text{double Poisson bracket } \{\!\{-,-\}\!\} \\ \text{NC moment map } \mu\end{array} & \dashrightarrow & \begin{array}{c}\text{comm. algebra } \mathbb{C}[\operatorname{Rep}(\mathcal{A},n)] \\ \text{Poisson bracket } \{-,-\}_{\mathrm{P}} \\ \text{moment map } (\mu_{ij})\end{array} \\
\mathcal{A}^\lambda := \mathcal{A}/(\mu-\lambda) \Big\downarrow \lambda\in\mathbb{C} & & \text{Ham. red.} \Big\downarrow \text{at } \lambda\,\mathrm{Id}_n \\
\begin{array}{c}\text{vector space } \mathcal{A}^\lambda/[\mathcal{A}^\lambda,\mathcal{A}^\lambda] \\ \text{Lie bracket } m\circ\{\!\{-,-\}\!\}\end{array} & \dashrightarrow & \begin{array}{c}\text{comm. algebra } \mathbb{C}[\operatorname{Rep}(\mathcal{A}^\lambda,n)]^{\mathrm{GL}_n} \\ \text{Poisson bracket } \{-,-\}_{\mathrm{P}}\end{array}
\end{array}$$

(cartoon in type $\mathtt{A}$)

Plan for the talk

1. **NC Poisson geometry (after Van den Bergh, Olshanski-Safonkin)**
2. More results in NC Poisson geometry
3. NC quasi-Poisson geometry (after Van den Bergh, and more)

Double (Poisson) brackets

$\mathcal{A}$ is a unital associative fin. gen. algebra over $\mathbb{C}$, $\otimes := \otimes_{\mathbb{C}}$

Definition ([Van den Bergh,'08])

A double bracket is a $\mathbb{C}$-linear map $\{\!\{-,-\}\!\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}^{\otimes 2}$ s.t.

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (skewsymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1) \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a) \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} (d \otimes 1)$ (inner derivation)

A double bracket is Poisson if "double Jacobi" $\mathbb{D}\mathrm{Jac} : \mathcal{A}^{\otimes 3} \to \mathcal{A}^{\otimes 3}$ vanishes.

$$\mathbb{D}\mathrm{Jac}(a,b,c) = \{\!\{a, \{\!\{b,c\}\!\}\}\!\}_L + \tau_{(123)} \{\!\{b, \{\!\{c,a\}\!\}\}\!\}_L + \tau_{(132)} \{\!\{c, \{\!\{a,b\}\!\}\}\!\}_L$$

Definition ([Van den Bergh,'08])

$\mu \in \mathcal{A}$ is a moment map if $\{\!\{\mu, a\}\!\} = a \otimes 1 - 1 \otimes a$ for all $a \in \mathcal{A}$

DPA : examples

Key property: enough to define $\{\!\{-,-\}\!\}$ on generators

Example

Take $\mathcal{A}_1 = \mathbb{C}[x]$ (or any quotient $\mathbb{C}[x]/(x^k)$).
$\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$
$\mu = x \in \mathcal{A}_1$

Example

Take $\mathcal{A}_2 = \mathbb{C}\langle x,y\rangle$
$\{\!\{x,y\}\!\} = 1 \otimes 1$, $\{\!\{x,x\}\!\} = 0$, $\{\!\{y,y\}\!\} = 0$
$\mu = [x,y] \in \mathcal{A}_2$

Relation to Kontsevich-Rosenberg principle

$\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]$ generated by symbols a_{ij}, $\forall a \in \mathcal{A}$, $1 \leq i,j \leq n$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Theorem (Van den Bergh,'08)

If $\mathcal{A}$ has a double bracket $\{\!\{-,-\}\!\}$, then for any $n \geq 1$ $\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]$ has an antisymmetric biderivation $\{-,-\}_n$ defined by

$$\{a_{ij}, b_{kl}\}_n = \{\!\{a,b\}\!\}'_{kj} \ \{\!\{a,b\}\!\}''_{il} \ .$$

If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}_n$ is a Poisson bracket.
Furthermore, $\mu \in \mathcal{A}$ moment map $\Rightarrow$ (μ_{ij}) is "geometric" moment map.

Proof.

$$\mathrm{Jac}(a_{ij}, b_{kl}, c_{uv}) = \mathbb{D}\mathrm{Jac}(a,b,c)_{uj,il,kv} - \mathbb{D}\mathrm{Jac}(a,c,b)_{kj,iv,ul}$$ □

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$\Rightarrow$ Set $\mathrm{tr}(c) := \sum_{1 \leq j \leq n} c_{jj}$. Then $\{\mathrm{tr}(a), \mathrm{tr}(b)\}_n = \mathrm{tr}(\mathrm{m} \circ \{\!\{a,b\}\!\})$ defines a Poisson bracket on the invariant subalgebra $\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]^{\mathrm{GL}_n}$.

DPA : examples and representations

Example

Take $\mathcal{A}_1 = \mathbb{C}[x]$ (or any quotient $\mathbb{C}[x]/(x^k)$).
$\{\!\{x, x\}\!\} = x \otimes 1 - 1 \otimes x$
Induces PB of $\mathfrak{gl}_n \simeq \mathbb{C}[\mathrm{Rep}(\mathcal{A}_1, n)]$ (or k-nilpotent matrices)

Example

Take $\mathcal{A}_2 = \mathbb{C}\langle x, y\rangle$
$\{\!\{x, y\}\!\} = 1 \otimes 1$, $\{\!\{x, x\}\!\} = 0$, $\{\!\{y, y\}\!\} = 0$
Induces PB of $T^*\mathfrak{gl}_n \simeq \mathbb{C}[\mathrm{Rep}(\mathcal{A}_2, n)]$

$\rightsquigarrow$ Second example extends to any path algebra of quiver !

Olshanski-Safonkin's idea

How to replace the action of GL_n on $\mathrm{Rep}(\mathcal{A}, n)$ with
– orthogonal group action? (type $\mathtt{B}, \mathtt{D}$)
– symplectic group action? (type $\mathtt{C}$)

Need an $i)$ anti-involution and $ii)$ bracket-compatibility [Olshanski-Safonkin,'24]

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Remark

- Their approach is basis-free.
  It will be illustrated for $\mathrm{O}_n \subset \mathrm{GL}_n$ realized in the usual way
- Inducing a moment map is due to [Safonkin, private comm.]
- Their results are not stated in the semi-simple case
  (where $1 = \sum_s e_s \in \mathcal{A}$ decomp. in orthog. idempotents)
  but the statements hold as well [Arthamonov-F.,'26]
- In the semi-simple case, one can mix copies of O_{n_s} and Sp_{n_t}

Preliminaries for [OS]

Definition

$(\mathcal{A},\phi)$ is an involutive algebra if $\phi:\mathcal{A}\to\mathcal{A}$ is a $\mathbb{C}$-linear anti-automorphism such that $\phi\circ\phi=\mathrm{id}_{\mathcal{A}}$.

We say that $\{\!\!\{-,-\}\!\!\}$ is ϕ-adapted, or that ϕ is bracket-compatible, if

$$\phi^{\otimes 2}(\{\!\!\{a,b\}\!\!\})=\tau_{(12)}\,\{\!\!\{\phi(a),\phi(b)\}\!\!\}\qquad \forall a,b\in\mathcal{A}.$$

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$$\phi^{\otimes 2}(\{\!\{a,b\}\!\}) = \tau_{(12)}\,\{\!\{\phi(a),\phi(b)\}\!\} \qquad \forall a,b \in \mathcal{A}.$$

Let τ be the matrix transposition $\xi \mapsto \xi^T$ on $\mathfrak{gl}_n$ (orthogonal type).

Definition

The twisted representation space is
$$\mathrm{Rep}^{\phi,\tau}(\mathcal{A},n) = \{\rho \in \mathrm{Rep}(\mathcal{A},n) \mid \rho \circ \phi = \tau \circ \rho\}.$$

For the coordinate ring, $\mathbb{C}[\mathrm{Rep}^{\phi,\tau}(\mathcal{A},n)] = \mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]/I^{\phi,\tau}$ with

$$I^{\phi,\tau} := \langle \phi(a)_{ij} - a_{ji} \,|\, a \in \mathcal{A},\ 1 \leq i,j \leq N \rangle$$

It suffices to check on generators: $\phi^{\otimes 2}(\{\!\{a,b\}\!\}) = \tau_{(12)}\,\{\!\{\phi(a),\phi(b)\}\!\}$

Example

Take $\mathcal{A}_1 = \mathbb{C}[x]$, $\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$ with $\phi(x) = -x$

$$\mathrm{Rep}^{\phi,\tau}(\mathcal{A}_1, n) = \{X \in \mathfrak{gl}_n \mid X^T = -X\}$$

Example

Take $\mathcal{A}_2 = \mathbb{C}\langle x,y\rangle$, $\{\!\{x,y\}\!\} = 1 \otimes 1$, $\{\!\{x,x\}\!\} = 0$, $\{\!\{y,y\}\!\} = 0$

with $\phi(x) = \zeta x$ and $\phi(y) = \zeta y$ ($\zeta \in \{\pm 1\}$)

$$\mathrm{Rep}^{\phi,\tau}(\mathcal{A}_2, n) \simeq T^*\{X \in \mathfrak{gl}_n \mid X^T = \zeta X\}$$

Main theorem [OS]

Theorem (Orthogonal case, [Olshanski-Safonkin,'24], [Safonkin, private comm.])

Let $(\mathcal{A}, \{\!\{-,-\}\!\})$ be a double Poisson algebra, ϕ be involutive and bracket-compatible (with $\{\!\{-,-\}\!\}$). The skewsymmetric biderivation $\{-,-\}_{\phi,\mathrm{O}}$ on $\mathrm{Rep}^{\phi,\tau}(\mathcal{A},n)$ uniquely defined by

$$\{a_{ij}, b_{kl}\}_{\phi,\mathrm{O}} = \frac{1}{2}\{\!\{a,b\}\!\}_{kj,il} + \frac{1}{2}\{\!\{\phi(a),b\}\!\}_{ki,jl}\,, \quad a,b \in \mathcal{A},$$

is a Poisson bracket for the action of O_n.
Furthermore, if $\mu \in \mathcal{A}$ is a moment map, then $\mathtt{X}(\mu) : \mathrm{Rep}^{\phi,\tau}(\mathcal{A},n) \to \mathfrak{o}_n$ is a moment map (up to adding a constant to μ).

Remark

On trace functions: $\{\mathrm{tr}(a), \mathrm{tr}(b)\}_{\phi,\mathrm{O}} = \frac{1}{2}\,\mathrm{tr}(\mathrm{m} \circ (\{\!\{a,b\}\!\} + \{\!\{\phi(a),b\}\!\}))$

Proof of the main theorem [OS]

Proof.

Recall that $\{a_{ij}, b_{kl}\}_{\phi,\mathrm{O}} = \frac{1}{2}\,\{\!\{a, b\}\!\}_{kj,il} + \frac{1}{2}\,\{\!\{\phi(a), b\}\!\}_{ki,jl}$.

$\square$

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Recall that $\{a_{ij}, b_{kl}\}_{\phi,\mathrm{O}} = \frac{1}{2}\{\!\{a,b\}\!\}_{kj,il} + \frac{1}{2}\{\!\{\phi(a),b\}\!\}_{ki,jl}$.

Main part: $\mathrm{Jac}_{\{-,-\}_{\phi,\mathrm{O}}}(a_{ij}, b_{kl}, c_{uv})$ equals

$$\begin{aligned}
&\frac{1}{4}\mathbb{D}\mathrm{Jac}(a,b,c)_{uj,il,kv} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(a,c,b)_{kj,iv,ul} \\
&+\frac{1}{4}\mathbb{D}\mathrm{Jac}(\phi(a),b,c)_{ui,jl,kv} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(\phi(a),c,b)_{ki,jv,ul} \\
&+\frac{1}{4}\mathbb{D}\mathrm{Jac}(a,\phi(b),c)_{uj,ik,lv} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(a,c,\phi(b))_{lj,iv,uk} \\
&+\frac{1}{4}\mathbb{D}\mathrm{Jac}(a,b,\phi(c))_{vj,il,ku} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(a,\phi(c),b)_{kj,iu,vl}\,.
\end{aligned}$$

For moment map property: by bracket-compatibility, μ and $-\phi(\mu)$ are moment maps, thus $\mu+\phi(\mu)=0$ (up to constant). Hence $\mathtt{X}(\mu)\in\mathfrak{o}_\alpha$, and conclude by direct computation. □

The new picture

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{Algebra } \mathcal{A} \\ \text{double Poisson bracket } \{\!\{-,-\}\!\} \\ \text{NC moment map } \mu\text{, compatible } \phi\end{array} \arrow[r, dashed] \arrow[dd, "\mathcal{A}^\lambda := \mathcal{A}/(\mu-\lambda)"', "\lambda \in \mathbb{C}"] & \begin{array}{c}\text{comm. algebra } \mathbb{C}[\mathrm{Rep}^{\phi,\tau}(\mathcal{A},n)] \\ \text{Poisson bracket } \{-,-\}_{\phi,\mathrm{O}} \\ \text{moment map } (\mu_{ij})\end{array} \arrow[d, thick] \\
 & \begin{array}{c}\text{comm. algebra } \mathbb{C}[\mathrm{Rep}^{\phi,\tau}(\mathcal{A}^\lambda,n)]^{\mathrm{O}_n} \\ \text{Poisson bracket } \{-,-\}_{\phi,\mathrm{O}}\end{array} \\
\begin{array}{c}\text{vector space } \mathcal{A}^\lambda/[\mathcal{A}^\lambda,\mathcal{A}^\lambda] \\ \text{Lie bracket}\end{array} \arrow[r, dashed] & \begin{array}{c}\text{trace algebra } \mathbb{C}[\mathrm{Rep}^{\phi,\tau}(\mathcal{A}^\lambda,n)]^{\mathrm{tr}} \\ \text{Poisson bracket } \{-,-\}_{\phi,\mathrm{O}}\end{array} \arrow[u, hook]
\end{tikzcd}
$$

About the symplectic case

Remark

The same story works in the symplectic case for the induced Poisson bracket:

For the dimension $2n$ (symmetry Sp_{2n}), one has

$$\begin{aligned}
\{a_{ij}, b_{kl}\}_{\phi,\mathrm{Sp}} &= \frac{1}{2}\,\{\!\{a,b\}\!\}_{kj,il} - \frac{1}{2}\,\{\!\{\phi(a),b\}\!\}_{k(i+n),(j+n)l}\,, &&\text{for } 1 \leq i,j \leq n,\\
&= \frac{1}{2}\,\{\!\{a,b\}\!\}_{kj,il} - \frac{1}{2}\,\{\!\{\phi(a),b\}\!\}_{k(i-n),(j-n)l}\,, &&\text{for } n+1 \leq i,j \leq 2n,\\
&= \frac{1}{2}\,\{\!\{a,b\}\!\}_{kj,il} + \frac{1}{2}\,\{\!\{\phi(a),b\}\!\}_{k(i+n),(j-n)l}\,, &&\text{for } n+1 \leq i \leq 2n,\ 1 \leq j \leq n,\\
&= \frac{1}{2}\,\{\!\{a,b\}\!\}_{kj,il} + \frac{1}{2}\,\{\!\{\phi(a),b\}\!\}_{k(i-n),(j+n)l}\,, &&\text{for } 1 \leq i \leq n,\ n+1 \leq j \leq 2n.
\end{aligned}$$

Plan for the talk

1. NC Poisson geometry (after Van den Bergh and Olshanski-Safonkin)
2. **More results in NC Poisson geometry**
3. NC quasi-Poisson geometry (after Van den Bergh, and more)

From now on, results without citations are from [Arthamonov-F., arXiv:2605.23369]

Quivers and NC cotangent spaces

The previous formalism can be applied to quivers.

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Quivers and NC cotangent spaces

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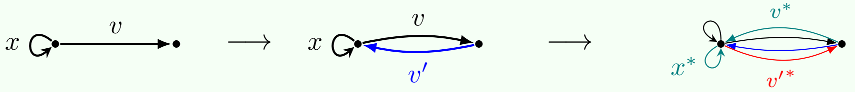
Example



Υ $\qquad Q_{\Upsilon,x}$ $\qquad \overline{Q}_{\Upsilon,x}$

$\phi(x) = -x$, $\phi(v) = v'$, $\phi(v^*) = v'^*$

$$\mathrm{Rep}^{\phi,\tau}(\mathbb{C}\overline{Q}_{\Upsilon,x},(n,1)) \simeq \{X, Y \in \mathfrak{o}_n,\ V \in \mathbb{C}^n,\ W \in (\mathbb{C}^n)^*\}$$

has Poisson structure with moment map $\mathcal{X}(\mu) = [X,Y] + VW + (VW)^T$.

By symplectic reduction, get $\mathcal{X}(\mu)^{-1}(0)/\!/\,\mathrm{O}_n$ with Poisson structure.

$\rightsquigarrow$ rank one version of *commuting variety* of O_n

Quiver varieties in the orthogonal case

For a second application, let Q be an *arbitrary* quiver.
Fix dimension vector $\alpha = (\alpha_s) \in \mathbb{N}^I$ indexed by vertices.
Put $\mathrm{O}_\alpha = \prod_s \mathrm{O}_{\alpha_s} \subset \prod_s \mathrm{GL}_{\alpha_s}$

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Corollary

The Poisson structure $\{-,-\}_{\phi,\mathrm{O}}$ *on* $\mathrm{Rep}(\mathbb{C}\overline{Q},\alpha)$ *obtained by Van den Bergh is* O_α*-equivariant.*
Furthermore, $\mathtt{X}(\mu_{\mathfrak{o}}) : \mathrm{Rep}(\mathbb{C}\overline{Q},\alpha) \to \mathfrak{o}_\alpha$ *is a moment map for*
$\mu_{\mathfrak{o}} = \sum_{a\in Q} \big([\mathtt{X}(a),\mathtt{X}(a^*)] + [\mathtt{X}(a)^T,\mathtt{X}(a^*)^T] \big)$

Orthogonal quiver variety: $\mathtt{X}(\mu_{\mathfrak{o}})^{-1}(0)/\!/\,\mathrm{O}_\alpha$ (or add stability parameters)

Sad news: can't reduce outside 0 at NC level...

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$\mu_{\mathfrak{o}} = \sum_{a\in Q} \left([\mathtt{X}(a),\mathtt{X}(a^*)] + [\mathtt{X}(a)^T,\mathtt{X}(a^*)^T] \right)$

Proof in NC geometry.

1. Pass to $A_Q = \mathbb{C}Q * \mathbb{C}Q^{op}$ with $\phi_0(a) = a^{op}$.
2. Make cotangent lift $\mathcal{A} = \mathbb{T}^* A_Q$ of A_Q
3. $\mathrm{Rep}^{\phi,\tau}(\mathcal{A},\alpha) \simeq \mathrm{Rep}(\mathbb{C}\overline{Q},\alpha)$. □

Orthogonal quiver variety: $\mathtt{X}(\mu_{\mathfrak{o}})^{-1}(0)/\!/\,\mathrm{O}_\alpha$ (or add stability parameters)

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More for quivers...

We can also "mix types".
WARNING: ϕ is only involutive 'up to sign' although $\phi^4 = \mathrm{id}_{\mathcal{A}}$

Example

For $\mathtt{A}_{2\ell}$ with alternating types O and Sp: recover (complex versions of) master phase spaces for orthosymplectic gauge theories [Gaiotto-Witten,'09]

WARNING: we can not reproduce the whole *geometric* formalism of [Y. Li,'19], [Nakajima,'25]. We only have a NC map ϕ

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Double quasi-Poisson brackets

Definition

A double bracket on $\mathcal{A}$ is quasi-Poisson if

$$\mathbb{D}\mathrm{Jac}(a,b,c) = \frac{1}{4}(ca \otimes b \otimes 1 - ca \otimes 1 \otimes b - c \otimes ab \otimes 1 + c \otimes a \otimes b$$
$$- a \otimes b \otimes c + a \otimes 1 \otimes bc + 1 \otimes ab \otimes c - \otimes a \otimes bc)\,.$$

Definition

An element $\Phi \in \mathcal{A}^\times$ is a (multiplicative) moment map if
$$\{\!\{\Phi, a\}\!\} = \tfrac{1}{2}(a \otimes \Phi - 1 \otimes \Phi a + a\Phi \otimes 1 - \Phi \otimes a) \quad \forall a \in \mathcal{A}$$

Relation to Kontsevich-Rosenberg principle

Theorem (Van den Bergh,'08)

If $\mathcal{A}$ has a double quasi-Poisson bracket $\{\!\{-,-\}\!\}$, then for any $n \geq 1$ $\mathbb{C}[\mathrm{Rep}(\mathcal{A},n)]$ has a quasi-Poisson bracket $\{-,-\}_n$ defined by

$$\{a_{ij}, b_{kl}\}_n = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}\ .$$

Furthermore, $\Phi \in \mathcal{A}^\times$ moment map $\Rightarrow$ (Φ_{ij}) is "geometric" moment map.

$$\begin{tikzcd}
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\end{tikzcd}$$

Double quasi-Poisson brackets and compatibility

Recall, if $(\mathcal{A}, \phi)$ is an *involutive algebra*, ϕ is *bracket-compatible*, if one has $\phi^{\otimes 2}(\{\!\{a, b\}\!\}) = \tau_{(12)} \{\!\{\phi(a), \phi(b)\}\!\}$, $\forall a, b \in \mathcal{A}$ generators

View $L_{g,r} = \mathbb{C}\langle x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_g^{\pm 1}, y_g^{\pm 1}, z_1^{\pm 1}, \ldots, z_r^{\pm 1}\rangle$ as $\mathbb{C}\pi_1(\Sigma_{g,r+1}, *)$.
It has a double quasi-Poisson bracket [Massuyeau-Turaev,'14]

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It has a double quasi-Poisson bracket [Massuyeau-Turaev,'14]

Example $((g = 0, r = 1))$

On $L_{0,1} = \mathbb{C}\langle z^{\pm 1}\rangle$, we have $\{\!\{z, z\}\!\} = \frac{1}{2}(z^2 \otimes 1 - 1 \otimes z^2)$, $\Phi = z$.
$\phi : L_{0,1} \to L_{0,1}$ satisfying $z \mapsto z^{-1}$ is bracket-compatible.

Example $((g = 1, r = 0))$

On $L_{1,0} = \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}\rangle$, we have $\Phi = [x, y]_{grp}$ and
$\{\!\{x, x\}\!\} = \frac{1}{2}(x^2 \otimes 1 - 1 \otimes x^2)$, $\quad \{\!\{y, y\}\!\} = -\frac{1}{2}(y^2 \otimes 1 - 1 \otimes y^2)$,
$\{\!\{x, y\}\!\} = \frac{1}{2}(yx \otimes 1 + 1 \otimes xy - x \otimes y + y \otimes x)$.
$\phi : L_{1,0} \to L_{1,0}$ satisfying $x \mapsto x^{-1}$, $y \mapsto y^{-1}$, is bracket-compatible.

Same for general (g, r) because "**bracket-compatibility is stable by fusion**"

"Main" new theorem

Theorem (Orthogonal case)

Let $(\mathcal{A}, \{\!\{-,-\}\!\})$ *be a double quasi-Poisson algebra,* ϕ *be involutive and bracket-compatible (with* $\{\!\{-,-\}\!\}$*). The skewsymmetric biderivation* $\{-,-\}_{\phi,\mathrm{O}}$ *on* $\mathrm{Rep}^{\phi,\tau}(\mathcal{A},n)$ *uniquely defined by*

$$\{a_{ij}, b_{kl}\}_{\phi,\mathrm{O}} = \frac{1}{2}\,\{\!\{a,b\}\!\}_{kj,il} + \frac{1}{2}\,\{\!\{\phi(a),b\}\!\}_{ki,jl}\,, \quad a,b\in\mathcal{A},$$

is a quasi-Poisson bracket for the action of O_n*.*
Furthermore, if $\Phi\in\mathcal{A}^\times$ *is a moment map, then* $\mathtt{X}(\Phi):\mathrm{Rep}^{\phi,\tau}(\mathcal{A},n)\to\mathrm{O}_n$ *is a moment map (up to multiplying* Φ *by a constant so that* $\Phi\phi(\Phi)=1$*).*

Proof of the main new theorem

Proof.

As in Poisson case, $\mathrm{Jac}_{\{-,-\}_{\phi,\mathrm{O}}}(a_{ij}, b_{kl}, c_{uv})$ equals

$$\begin{aligned}
&\frac{1}{4}\mathbb{D}\mathrm{Jac}(a,b,c)_{uj,il,kv} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(a,c,b)_{kj,iv,ul} \\
&+\frac{1}{4}\mathbb{D}\mathrm{Jac}(\phi(a),b,c)_{ui,jl,kv} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(\phi(a),c,b)_{ki,jv,ul} \\
&+\frac{1}{4}\mathbb{D}\mathrm{Jac}(a,\phi(b),c)_{uj,ik,lv} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(a,c,\phi(b))_{lj,iv,uk} \\
&+\frac{1}{4}\mathbb{D}\mathrm{Jac}(a,b,\phi(c))_{vj,il,ku} - \frac{1}{4}\mathbb{D}\mathrm{Jac}(a,\phi(c),b)_{kj,iu,vl}\,.
\end{aligned}$$

HARD part:

1. use the explicit form of $\mathbb{D}\mathrm{Jac} \rightsquigarrow 8\times 8$ terms
2. Identify them with the action of the Cartan trivector of O_n

For the moment map: multiplicative version of the Poisson case using that Φ and $\phi(\Phi)^{-1}$ are moment maps. □

For character varieties

Example

Taking again $L_{g,r} \simeq \mathbb{C}\pi_1(\Sigma_{g,r+1}, *)$ with same ϕ, the formula reproduces the construction of [Massuyeau-Turaev,'18].

$\rightsquigarrow \mathtt{X}(\Phi)^{-1}(1)/\!/\,\mathrm{O}_n$ is the O_n character variety with genus g and r punctures

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Some remarks:

- They use another technique: Hopf algebra + Fox pairing
  Our formalism encompasses their framework (in the ungraded case)
- We can easily write down the symplectic case (they did not do this explicitly)

Quivers in the quasi-Poisson case

One has a "groupoid" analogue of the case $\mathbb{C}\pi_1(\Sigma_{g,r+1},*)$ based on quivers It requires the dimension vector $\alpha := n^I = (n,\dots,n)$.

Corollary

The quasi-Poisson structure $\{-,-\}_\alpha$ defined on an open subspace of $\mathrm{Rep}(\mathbb{C}\overline{Q},\alpha)$ obtained by Van den Bergh can be modified into a quasi-Poisson bracket $\{-,-\}_{\phi,\mathrm{O}}$ for the underlying O_α-action on

$$\mathrm{O}_n^{2|Q|} \simeq \mathrm{Rep}^{\phi,\tau}(A_{Q,0},\alpha) \subset \mathrm{Rep}(\mathbb{C}\overline{Q},\alpha)$$

Furthermore, $\mathtt{X}(\Phi) : \mathrm{Rep}^{\phi,\tau}(A_{Q,0},\alpha) \to \mathrm{O}_\alpha$ is a moment map for

$$\mathtt{X}(\Phi) = \overrightarrow{\prod_{a\in Q}} (1_a + \mathtt{X}(a)\mathtt{X}(a^*))(1_{a^*} + \mathtt{X}(a^*)\mathtt{X}(a))^{-1}$$

Orthogonal mult. quiver variety: $\mathtt{X}(\Phi)^{-1}(1)/\!/\,\mathrm{O}_\alpha$ (or add stability param.)

1_a is $\mathrm{Id}_{n|I|}$ from which we remove the diagonal block Id_n in position $t(a)$

Quivers in the quasi-Poisson case (application)

We follow [Maiza,'26]. Assume the quiver has no *isolated vertex*. Put

$$I_{int} = \{ s \in I \mid |h^{-1}(s)| \neq 1 \text{ and } |t^{-1}(s)| \neq 1 \} .$$

Consider the action of $G_{int} := \prod_{s \in I_{int}} \mathrm{O}_n \hookrightarrow \prod_{s \in I} \mathrm{O}_n$ on $\mathrm{Rep}^{\phi,\tau}(A_{Q,0}, \alpha)$, and the corresponding G_{int}-valued factor $\mathtt{X}(\Phi)_{int} := \prod_{s \in I_{int}} \mathtt{X}(\Phi_s)$ of the moment map. Then, we set

$$\mathcal{N}_G(\Upsilon) := (\mathtt{X}(\Phi)_{int})^{-1}(1) /\!\!/ G_{int} .$$

These are complex-valued versions of spaces used in [Maiza,'26] to build (quasi-Hamiltonian space)-valued TQFTs.

Another application: Kontsevich system

[Arthamonov,'17]: double quasi-Poisson interpretation on $A := \mathbb{C}\langle u^{\pm 1}, v^{\pm 1}\rangle$ of Kontsevich system from [Wolf-Efimovskaya,'10]

$$\frac{\mathrm{d}u}{\mathrm{d}t} = uv - uv^{-1} - v^{-1}, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = vu^{-1} - vu + u^{-1}$$

$\rightsquigarrow$ "integrable" dynamics on quasi-Poisson double of GL_n + reductions.
BUT : lose involutivity of the family for O_n and Sp_n

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For any fixed $\lambda \in \mathbb{C}$, consider the modified system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = uv - uv^{-1} - \lambda v^{-1} + \lambda uvu, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = vu^{-1} - vu + \lambda u^{-1} - \lambda vuv$$

Key remark: Hamiltonian invariant under anti-invol. $\phi : u \mapsto u^{-1}$, $v \mapsto v^{-1}$
$h_\lambda := u + v + u^{-1} + v^{-1} + \lambda(u^{-1}v^{-1} + vu)$
$\rightsquigarrow$ "integrable" dynamics on qPoisson double of $\mathrm{GL}_n, \mathrm{O}_n, \mathrm{Sp}_n$ + reds.

Thank you for your attention !

Maxime Fairon
maxime.fairon@u-bourgogne.fr • maxime.fairon@ube.fr
mfairon.perso.math.cnrs.fr